

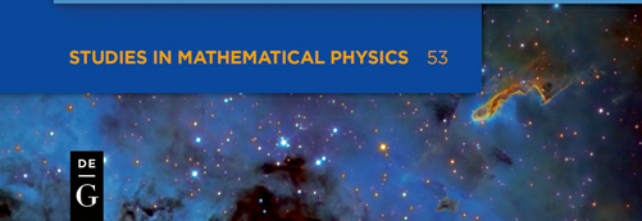
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Vladimir K. Dobrev

INVARIANT DIFFERENTIAL OPERATORS

VOLUME 4: ADS/CFT, (SUPER-)VIRASORO, AFFINE
(SUPER-)ALGEBRAS

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Vladimir K. Dobrev

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Vladimir K. Dobrev

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Preface

This is Volume 4 of our monograph series on invariant differential operators. In Volume 1 we presented our canonical procedure for the construction of invariant differential operators and showed its application to the objects of the initial domain—noncompact semisimple Lie algebras and groups. In Volume 2 we gave a detailed exposition with many concrete examples of the application of our procedure to *quantum groups*. In Volume 3 we gave a detailed exposition with many concrete examples of the application of our procedure to *supersymmetry*.

Chapter 1 of Volume 4 treats aspects of the AdS/CFT correspondence, or, more generally, of relativistic and nonrelativistic holography. First we present the intertwining operator realization of the Euclidean AdS/CFT correspondence. The symmetry object here is the Euclidean conformal (or de Sitter) group $SO(n, 1)$. Then we give a similar exposition for the anti de Sitter holography on the example of the anti de Sitter group $SO(3, 2)$. In the last section we treat nonrelativistic holography using as symmetry object the Schrödinger group (called also the nonrelativistic conformal group).

Chapter 2 treats in detail nonrelativistic invariant differential operators and equations. We start with the easiest example of invariant differential equations for the $(1 + 1)$ -dimensional space-time. Next we give the general expressions for the invariant differential equations in the case of $(n + 1)$ -dimensional space-time. Finally, we consider in great detail the case of $(3 + 1)$ -dimensional space-time as this is the most relevant case for the applications. Next we consider the q -deformed situation and give the construction of the q -Schrödinger algebra and the corresponding invariant operators and equations. Finally, we present difference analogues of the free Schrödinger equation.

Chapter 3 treats in detail the Virasoro and super-Virasoro algebras, their representations, character formulae and modular invariants. We use a unified treatment for the Virasoro and $N = 1$ super-Virasoro algebras. A very important tool is the multiplet classification of the reducible (generalized) Verma modules over these (super-)algebras. Using the multiplet classification we derive the character formulae for all irreducible (generalized) highest weight modules. On the basis of the multiplet classification we also introduce a Weyl group for Virasoro and $N = 1$ super-Virasoro algebras. Another development is the consideration of the Fock modules over the Virasoro algebra using which we give new formulae for the singular vectors which are not accessible in the category of Verma modules.

Next we pass on the $N = 2$ super-Virasoro algebras. We give the characters of the unitarizable highest weight modules over the $N = 2$ superconformal algebras. Then we consider the related modular invariants for theta-functions with characteristics and the twisted $N = 2$ superconformal and $su(2)$ Kac–Moody algebras. Further we give the classification of modular invariant partition functions for the twisted $N = 2$ superconformal algebra, for the twisted $su(2)$ Kac–Moody algebra and D_{2k} parafermions.

Chapter 4 treats examples of affine Lie (super-)algebras. First we present the multiplet classification of Verma modules over affine Lie algebras and the invariant differential operators explicitly in the case of the affine algebra $A_\ell^{(1)}$. From the latter we use the case of $A_1^{(1)}$ to introduce new Weyl groups and to give the characters of singular highest weight modules over $A_1^{(1)}$. We also derive a correspondence between some representations of Virasoro and $A_1^{(1)}$ algebras. Next we consider the case of the $SO(3,2)$ Kac–Moody algebra for which we construct a special representation. Finally, we consider the multiplets of Verma modules over the $osp(2, 2)^{(1)}$ super-Kac–Moody algebra.

Each chapter has a summary which explains briefly the contents and the most relevant literature. Besides there is a bibliography, an author index, and a subject index.

Sofia, August 2018

Vladimir Dobrev

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1 Relativistic and nonrelativistic holography

Summary

The AdS/CFT correspondence was introduced by Maldacena 20 years ago [433]. Soon important contributions were made by Gubser–Klebanov–Polyakov [293] and by Witten [572]. We recall the two ingredients of the AdS/CFT correspondence [433, 293, 572]: 1. the holography principle, which is very old, and means the reconstruction of some objects in the bulk (which may be classical or quantum) from some objects on the boundary; 2. the reconstruction of quantum objects, like 2-point functions on the boundary, from appropriate actions on the bulk.

Here we give a group-theoretic interpretation of the AdS/CFT correspondence as a relation of a representation equivalence between representations of the conformal group describing the bulk AdS fields ϕ , their boundary fields ϕ_0 and the boundary conformal operators $\mathcal{O}$ coupled to the latter. We use two kinds of equivalences. The first kind is the equivalence between the representations describing the bulk fields and the boundary fields and it is established here. The second kind is the equivalence between conjugated conformal representations related by Weyl reflection, e. g. the coupled fields ϕ_0 and $\mathcal{O}$. Operators realizing the first kind of equivalence for special cases were actually given by Witten and others—here they are constructed in a more general setting from the requirement that they are intertwining operators. The intertwining operators realizing the second kind of equivalence are provided by the standard conformal two-point functions. Using both equivalences we find that the bulk field has in fact two boundary fields, namely, the coupled fields ϕ_0 and $\mathcal{O}$, the limits being governed by the corresponding conjugated conformal weights.

In this chapter we give a group-theoretic interpretation of relativistic holography as equivalence between representations in three cases:

1. the Euclidean conformal (or de Sitter) group;
2. the anti de Sitter group $SO(3,2)$;
3. the Schrödinger group.

In each case we give explicitly boundary-to-bulk operators and we show that these operators and the easier bulk-to-boundary operators are intertwining operators. Furthermore, we show that each bulk field has two boundary (shadow) fields with conjugated conformal weights. These fields are related by another intertwining operator, given by a two-point function on the boundary.

1.1 Intertwining operator realization of Euclidean AdS/CFT correspondence

1.1.1 Preliminaries

Recently there arose renewed interest in (super-)conformal field theories in arbitrary dimensions. This happened after the remarkable proposal in [433], according to which the large N limit of a conformally invariant theory in d dimensions is governed by supergravity (and string theory) on $d + 1$ -dimensional AdS space (often called AdS_{d+1}) times a compact manifold. Actually the possible relation of field theory on AdS_{d+1} to field theory on $\mathcal{M}_d$ has since long ago been a subject of interest; cf., e. g. [231, 22, 246,

473, 297, 183]. The proposal of [433] was elaborated in [293] and [572] where a precise correspondence was proposed between conformal field theory observables and those of supergravity: correlation functions in conformal field theory are given by the dependence of the supergravity action on the asymptotic behaviour at infinity. More explicitly, a conformal field $\mathcal{O}$ corresponds to an AdS field ϕ when there exists a conformal invariant coupling $\int \phi_0 \mathcal{O}$ where ϕ_0 is the value of ϕ at the boundary of AdS_{d+1} [572]. Furthermore, the dimension Δ of the operator $\mathcal{O}$ is given by the mass of the particle described by ϕ in supergravity [572]. Also the spectrum of Kaluza–Klein excitations of $\text{AdS}_5 \times S^5$ can be matched precisely with certain operators of the $N = 4$ super Yang–Mills theory in four dimensions [572]. After these initial papers there was an explosion of related research of which of interest to us are two aspects: 1) Calculation of conformal correlators from AdS (super-)gravity; cf., e. g. [311, 242, 426, 111, 458, 272, 411, 122, 562, 337, 68, 77, 72, 423, 136]. 2) Matching of gravity and string spectra with conformal theories; cf., e. g. papers by Ferrara, Flato, Fronsdal and others [223, 231, 486, 20, 225, 34, 222, 227, 482, 295].

The main aim of this chapter is to give a group-theoretic interpretation of the above correspondence. In fact such an interpretation is partially present in [183] for the $d = 3$ Euclidean version of the AdS/CFT correspondence in the context of the construction of discrete series representations of the group $\text{SO}(4, 1)$ involving symmetric traceless tensors of arbitrary nonzero spin.

In short the essence of our interpretation is that the above correspondence is a relation of *representation equivalence* between the representations describing the fields ϕ , ϕ_0 and $\mathcal{O}$. There are actually two kinds of equivalences. The first kind is new (besides the example from [183] mentioned above) and was proved first in [168]—it is between the representations describing the bulk fields and the boundary fields. The second kind is well known—it is the equivalence between boundary conformal representations which are related by Weyl reflections, the representations here being the coupled fields ϕ_0 and $\mathcal{O}$.

In this section we follow [168]. Our interpretation means that the operators relating these fields are *intertwining operators* between (partially) equivalent representations. Operators giving the first kind of equivalence for special cases were actually given in, e. g. [572, 426, 458]. Here they are constructed in a more general setting from the requirement that they are intertwining operators. The operators giving the second kind of equivalence are provided by the standard conformal two-point functions (and the latter intertwining property has been known since a long time ago; cf. [397] and [183]). Using both equivalences we find that the bulk field has naturally *two* boundary fields, namely, the coupled fields ϕ_0 and $\mathcal{O}$, the limits being governed by the corresponding conjugated conformal weights $d - \Delta$ and Δ . Thus, we notice that from the point of view of the bulk-to-boundary correspondence the coupled fields ϕ_0 and $\mathcal{O}$ are generically treated *on an equal footing*.

On the AdS side the representations are realized on *de Sitter space* S (the Euclidean counterpart of AdS space), which is isomorphic to the coset G/K . What is also

very essential for our approach is that S is isomorphic (via the Iwasawa decomposition) also to the solvable product group $\tilde{N}A$, where $\tilde{N}$ is the abelian group of Euclidean translations (isomorphic to $\mathbb{R}^d$), A is the one-dimensional dilatation group. The isomorphism $S \cong \tilde{N}A$ and related ones are explicated in the next subsection. On the conformal side the representations are realized on $\tilde{N}$, and we use also the fact that the latter is locally isomorphic (via the Bruhat decomposition) to the coset G/MAN , (where N is the group of special conformal transformations). These representations, called elementary representations (ERs), are introduced next. Then we introduce the representations on de Sitter space. Next we give the bulk-to-boundary and boundary-to-bulk intertwining operators and discuss the difference between equivalence and partial equivalence. We also display the second limit of the bulk fields and we derive some further intertwining relations. From the latter we derive the relation to the Plancherel measure for G . We illustrate the intertwining relations by a commutative diagram.

1.1.2 de Sitter space from Iwasawa decomposition

As mentioned, the relation between the two pictures uses the fact that $\mathbb{R}^d$ is easily identified within the $d + 1$ -dimensional de Sitter space ($d \geq 2$). Indeed, de Sitter space may be parametrized as

$$\xi_{d+2}^2 - \sum_{\alpha=1}^{d+1} \xi_{\alpha}^2 = 1, \quad \xi_{d+2} \geq 1, \quad (1.1)$$

and the first d of the ξ_{α} may be taken as coordinates on $\mathbb{R}^d$.

The group-theoretic interpretation of this relation is present in [183] using the so-called *Iwasawa decomposition* $G = \tilde{N}AK$.¹ This is a global decomposition, i. e. each element g of G can be decomposed as the product of the corresponding matrices: $g = \tilde{n}ak$, with $\tilde{n} \in \tilde{N}$, $a \in A$, $k \in K$. To be explicit we use the following defining relation of G :

$$G = \{g \in \text{GL}(d+2, \mathbb{R}) \mid {}^t g \eta g = \eta \doteq \text{diag}(-1, \dots, -1, 1), \\ \det g = 1, g_{d+2, d+2} \geq 1\}, \quad (1.2)$$

which is the identity component of $O(d+1, 1)$, (${}^t g$ is the transposed of g). Then we have the following matrix representations of the necessary subgroup elements (cf. formu-

¹ There are several versions of the Iwasawa decomposition, e. g. one may use the group N instead of $\tilde{N}$, and one may take a different order of the three factors involved. The choice of version is a matter of convenience.

lae (1.20a), (1.21), and (1.23), of [183], with $2h$ replaced here by d):

$$\tilde{n} = \tilde{n}_x = \begin{pmatrix} \delta_{ij} & -x_i & x_i \\ x_j & 1 - \frac{1}{2}x^2 & \frac{1}{2}x^2 \\ x_j & -\frac{1}{2}x^2 & 1 + \frac{1}{2}x^2 \end{pmatrix} \in \tilde{N},$$

$$x \in \mathbb{R}^d, \quad x^2 \equiv \sum_{i=1}^d x_i^2, \quad (1.3a)$$

$$a = \begin{pmatrix} \delta_{ij} & 0 & 0 \\ 0 & \frac{|a|^2+1}{2|a|} & \frac{|a|^2-1}{2|a|} \\ 0 & \frac{|a|^2-1}{2|a|} & \frac{|a|^2+1}{2|a|} \end{pmatrix} \in A, \quad |a| > 0, \quad (1.3b)$$

$$k = \begin{pmatrix} k_{ij} & k_{i,d+1} & 0 \\ k_{d+1,j} & k_{d+1,d+1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \in K, \quad (k_{\alpha\beta}) \in \text{SO}(d+1). \quad (1.3c)$$

Writing $g = \tilde{n}ak$ one may determine the factors $\tilde{n}$, a , k in terms of the matrix elements of g . We use this for the elements of the last column of g , which actually parametrize the de Sitter space, i. e.

$$\xi_A = g_{A,d+2}, \quad A = 1, \dots, d+2. \quad (1.4)$$

Indeed, substituting (1.4) in (1.1) we recover the $(d+2, d+2)$ -element of the defining relation (1.2), i. e.

$$({}^t g \eta g)_{d+2,d+2} = \eta_{d+2,d+2} \implies$$

$$g_{d+2,d+2}^2 - \sum_{a=1}^{d+1} g_{a,d+2}^2 = 1. \quad (1.5)$$

Now in terms of the parameters in (1.3) we get for the elements of the last column of g , respectively, for the parameters of de Sitter space

$$g_{i,d+2} = \xi_i = \frac{1}{|a|} x_i,$$

$$g_{d+1,d+2} = \xi_{d+1} = \frac{|a|^2 - 1 + x^2}{2|a|},$$

$$g_{d+2,d+2} = \xi_{d+2} = \frac{|a|^2 + 1 + x^2}{2|a|} \geq 1. \quad (1.6)$$

Notice that the only the $d+1$ parameters of the matrices $\tilde{n}_x$, a of the Iwasawa decomposition (cf. (1.3a,b)) are involved. Solving (1.6) we obtain for the latter parameters:

$$x_i = \frac{g_{i,d+2}}{g_{d+2,d+2} - g_{d+1,d+2}} = \frac{\xi_i}{\xi_{d+2} - \xi_{d+1}},$$

$$|a| = \frac{1}{g_{d+2,d+2} - g_{d+1,d+2}} = \frac{1}{\xi_{d+2} - \xi_{d+1}} > 0. \quad (1.7)$$

(The previous condition follows from $g_{d+2,d+2} \geq 1$.) From (1.6) we get also the consistency condition:

$$\frac{|a|^2 + x^2}{|a|} = g_{d+2,d+2} + g_{d+1,d+2} = \xi_{d+2} + \xi_{d+1}. \quad (1.8)$$

Indeed, inserting (1.7) in (1.8) we recover (1.1) and (1.5).

• Thus, in (1.6) and (1.7) we have the mentioned correspondence between de Sitter space S and the (coset) solvable subgroup $S = \tilde{N}A \cong G/K \cong \mathbb{R}^d \times \mathbb{R}_{>0}$ of the de Sitter group G . In addition, this explicates the group-theoretical interpretation of Euclidean space-time $\mathbb{R}^d$ as the abelian subgroup $\tilde{N}$ of the solvable subgroup S , and the topological interpretation of $\mathbb{R}^d$ as the boundary of $\mathbb{R}^d \times \mathbb{R}_{>0}$ for $|a| \rightarrow 0$.

Remark 1. Note that for *deven* some expressions are simpler if we work with the extended de Sitter group:

$$G' \doteq \{g \in \text{GL}(d+2, \mathbb{R}) \mid {}^t g \eta g = \eta, g_{d+2,d+2} \geq 1\}, \quad (1.9)$$

which includes reflections of the first $d+1$ axes. Then $K \rightarrow K' = O(d+1)$, $M \rightarrow M' = O(d)$, but the de Sitter space S and the results of this section are not changed. $\diamond$

1.1.3 Conformal field theory representations

The representations used in conformal field theory are called (in the representation theory of semisimple Lie groups) generalized principal series representations (cf. [390]). In [183, 185, 161] they were called *elementary representations* (ERs). They are obtained by induction from the subgroup $P = MAN$ (P is called a parabolic subgroup of G). The induction is from unitary irreps of $M = \text{SO}(d)$, from arbitrary (non-unitary) characters of A , and trivially from N . There are several realizations of these representations. We give now the so-called *noncompact picture* of the ERs—it is the one actually used in physics.

The representation space of these induced representations consists of smooth functions on $\mathbb{R}^d$ with values in the corresponding finite-dimensional representation space of M , i. e.

$$C_\chi = \{f \in C^\infty(\mathbb{R}^d, V_\mu)\} \quad (1.10)$$

where $\chi = [\mu, \Delta]$, Δ is the conformal weight, μ is a unitary irrep of M , V_μ is the finite-dimensional representation space of μ .

We use the standard $\text{SO}(p)$ representation parametrization of μ :

$$\mu = [\ell_1, \dots, \ell_{\tilde{p}}], \quad \tilde{p} \equiv \left\lfloor \frac{p}{2} \right\rfloor, \quad (1.11)$$

where all ℓ_j are simultaneously integer or half-integer, all are positive except for p even when ℓ_1 can also be negative, and they are ordered: $|\ell_1| \leq \ell_2 \leq \dots \leq \ell_{\tilde{p}}$.

Below we shall use also the *mirror-image* representation $\tilde{\mu}$ of μ . For d odd holds $\tilde{\mu} \equiv \mu$, while for $d \geq 4$ even $\tilde{\mu}$ may be obtained from μ by exchanging the representation labels of the two distinguished Dynkin nodes of $\text{SO}(d)$, i. e.

$$\tilde{\mu} = [-\ell_1, \ell_2, \dots, \ell_{\tilde{d}}]. \quad (1.12)$$

Analogously, for $d = 2$ we set

$$\tilde{\mu} = [-\ell_1]. \quad (1.13)$$

In addition, the functions (1.10) have a special asymptotic expansion as $x \rightarrow \infty$. The leading term of this expansion is $f(x) \sim \frac{1}{(x^2)^\Delta} f_0$ (for more details of this expansion we refer to [183, 185, 161]). The representation T^χ acts in C_χ by

$$(T^\chi(g)f)(x) = |a|^{-\Delta} \cdot \hat{d}^\mu(m)f(x') \quad (1.14)$$

where the nonglobal Bruhat decomposition $g = \tilde{n}man$ is used:²

$$g^{-1}\tilde{n}_x = \tilde{n}_{x'}m^{-1}a^{-1}n^{-1}, \quad g \in G, \tilde{n}_x, \tilde{n}_{x'} \in \tilde{N}, m \in M, a \in A, n \in N; \quad (1.15)$$

$\hat{d}^\mu(m)$ is the representation matrix of μ in V_μ .³

Remark 2. In the case $d = 2$ when $M = \text{SO}(2)$ and the elements of M are parametrized as

$$m_\theta = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix} \quad (1.16)$$

the elements of C_χ are

$$C_\chi = \{\varphi \in C^\infty(\mathbb{R}^2, \mathbb{C}), \varphi(z) = \varphi(x_1 + ix_2)\}. \quad (1.17)$$

The representation μ is parametrized by $\ell \in \mathbb{Z}$ and the representation $\hat{d}^\mu(m)$ acts as follows:

$$(\hat{d}^\mu(m_\theta)\varphi)(z) = e^{i\ell\theta}\varphi(e^{-i\theta}z). \quad (1.18)$$

² For the cases with measure zero for which $g^{-1}\tilde{n}_x$ does not have a Bruhat decomposition of the form $\tilde{n}man$ the action is defined separately, and the passage from (1.14) to these special cases is ensured to be smooth by the asymptotic properties mentioned above. Further, we may omit such measure-zero exceptions from the formulae—in a rigorous exposition all of them are taken care of; cf. [183, 185, 161].

³ One may interpret C_χ also as a space of smooth sections of the homogeneous vector bundle with base space G/MAN and fibre V_μ .

The action of the mirror image $\tilde{\mu}$ is

$$(\hat{d}^{\tilde{\mu}}(m_\theta)\varphi)(z) = e^{-i\ell\theta}\varphi(e^{-i\theta}z). \quad (1.19)$$

More details for the case $d = 2$ may be found in Appendix B of [183]. $\diamond$

The ERs are generically irreducible both operatorially (in the sense of Schur's lemma) and topologically (meaning nonexistence of nontrivial (closed) invariant subspaces). However, most of the physically relevant examples are when the ERs are topologically reducible and indecomposable. In particular, such are the representations describing gauge fields; cf., e. g. [494].

The importance of the elementary representations comes also from the remarkable result of Langlands–Knapp–Zuckerman [405, 391] stating that every irreducible admissible representation of a real connected semisimple Lie group G with finite centre is equivalent to a subrepresentation of an elementary representation of G .⁴ To obtain a subrepresentation of a topologically reducible ER one has to solve certain invariant differential equations, cf. [183, 185, 161].

Note that the representation data given by $\chi = [\mu, \Delta]$ fixes also the value of the Casimir operators $\mathcal{C}_i$ in the ER $\mathcal{C}_\chi$, independently of the latter reducibility. For later use we write

$$\mathcal{C}_i(\{X\})f(x) = \lambda_i(\mu, \Delta)f(x), \quad i = 1, \dots, \text{rank } G = \left\lfloor \frac{d}{2} \right\rfloor + 1, \quad (1.20)$$

where $\{X\}$ denotes symbolically the generators of the Lie algebra $\mathcal{G}$ of G , and the action of $X \in \mathcal{G}$ is given by the infinitesimal version of (1.14):

$$(Xf)(x) = \left. \frac{\partial}{\partial t} (T^X(\exp tX)f)(x) \right|_{t=0} \quad (1.21)$$

applying the Bruhat decomposition to $\exp(-tX)\tilde{n}_\chi$.

Next, we would like to recall the general expression of the conformal two-point function $G_\chi(x_1 - x_2)$ (for special cases cf. [495, 224, 238], for the general formula with special stress on the role of the conformal inversion; cf. [397] and [183]):

$$G_\chi(x) = \frac{y_\chi}{(x^2)^\Delta} \hat{d}^\mu(r(x)),$$

$$r(x) = \begin{pmatrix} \tilde{r}(x) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in M, \quad \tilde{r}(x) = \left(\frac{2}{x^2} x_i x_j - \delta_{ij} \right), \quad (1.22)$$

⁴ *Subrepresentations* are irreducible representations realized on invariant subspaces of the ER spaces (in particular, the irreducible ERs themselves). The admissibility condition is fulfilled in the physically interesting examples.

where γ_χ is an arbitrary constant for the moment. (Note that for d even $r(x) \in O(d)$, so we work with G' ; cf. (1.9).)

Finally, we note the intertwining property of $G_\chi(x)$. Namely, let $\tilde{\chi}$ be the representation conjugated to χ by Weyl reflection, i. e. by the nontrivial element of the two-element restricted Weyl group $W(G, A)$ [183]. Then we have

$$\tilde{\chi} \doteq [\tilde{\mu}, d - \Delta], \quad \text{for } \chi = [\mu, \Delta]. \quad (1.23)$$

Then we have the following intertwining operator [397, 183]:

$$G_\chi : C_{\tilde{\chi}} \longrightarrow C_\chi, \quad T^\chi(g) \circ G_\chi = G_\chi \circ T^{\tilde{\chi}}(g), \quad \forall g, \quad (1.24a)$$

$$(G_\chi f)(x_1) \doteq \int G_\chi(x_1 - x_2) f(x_2) dx_2. \quad (1.24b)$$

(Here $dx \equiv d^d x$.) This means that the representations are partially equivalent. Note that because of this equivalence all Casimir values coincide:

$$\lambda_i(\tilde{\mu}, d - \Delta) = \lambda_i(\mu, \Delta), \quad \forall i. \quad (1.25)$$

Note that at generic points the representations are equivalent, namely, one has [183, 185]:

$$G_\chi G_{\tilde{\chi}} = \mathbf{1}_\chi, \quad G_{\tilde{\chi}} G_\chi = \mathbf{1}_{\tilde{\chi}}. \quad (1.26)$$

This may be used to fix the constant γ_χ .

From the point of view of the AdS/CFT correspondence the importance of the pair $\chi, \tilde{\chi}$ is in the fact that the corresponding fields have a conformally invariant coupling through the standard bilinear form:

$$\langle \phi_0, \mathcal{O} \rangle \doteq \int dx \langle \phi_0(x), \mathcal{O}(x) \rangle_\mu, \quad \phi_0 \in C_{\tilde{\chi}}, \quad \mathcal{O} \in C_\chi, \quad (1.27)$$

where $\langle \cdot, \cdot \rangle_\mu$ is the standard pairing between μ and $\tilde{\mu}$. (Note that if ϕ_0 determines a p -form, (with $p < d$), then $\mathcal{O}(x)$ determines a $(d - p)$ -form.)

1.1.4 Representations on de Sitter space

In the previous section we discussed representations on $\mathbb{R}^d \cong \tilde{N}$ induced from the parabolic subgroup MAN which is natural since the abelian subgroup $\tilde{N}$ is locally isomorphic to the factor space G/MAN (via the Bruhat decomposition). Similarly, it is natural to discuss representations on de Sitter space $S \cong \tilde{N}A$ which are induced from the maximal compact subgroup $K = \text{SO}(d + 1)$ since the solvable group $\tilde{N}A$ is isomorphic to the factor space G/K (via the Iwasawa decomposition). Namely, we consider the representation space:

$$\hat{C}_\tau = \{\phi \in C^\infty(\mathbb{R}^d \times \mathbb{R}_{>0}, U_\tau)\} \quad (1.28)$$

where τ is an arbitrary unitary irrep of K , U_τ is the finite-dimensional representation space of τ , with representation action:

$$(\hat{T}^\tau(g)\phi)(x, |a|) = \tilde{D}^\tau(k)\phi(x', |a'|) \quad (1.29)$$

where the Iwasawa decomposition is used:

$$g^{-1}\tilde{n}_x a = \tilde{n}_{x'} a' k^{-1}, \quad g \in G, \quad k \in K, \quad \tilde{n}_x, \tilde{n}_{x'} \in \tilde{N}, \quad a, a' \in A \quad (1.30)$$

and $\tilde{D}^\tau(k)$ is the representation matrix of τ in U_τ . However, unlike the ERs, these representations are reducible, and to single out an irrep equivalent, say, a subrepresentation of an ER, one has to look for solutions of the eigenvalue problem related to the Casimir operators. This procedure is actually well understood and is used in the construction of the discrete series of unitary representations; cf. [325, 530] (also [183] for $d = 3$).

In the actual implementation of (1.29) it is convenient to use the unique decomposition

$$k = m(k)k_f, \quad m(k) = \begin{pmatrix} \tilde{m}(k) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in M, \quad k_f = \begin{pmatrix} \tilde{k}_f & 0 \\ 0 & 1 \end{pmatrix} \in K, \quad (1.31)$$

representing the decomposition of K into its subgroup M and the coset K/M : $K \cong M K/M$. Explicitly, using the parametrization of k in (1.3c), we have (for $k_{d+1,d+1} \neq -1$)

$$\begin{aligned} \tilde{m}(k) &= \left(k_{ij} - \frac{1}{1+k_{d+1,d+1}} k_{i,d+1} k_{d+1,j} \right), \\ \tilde{k}_f &= \begin{pmatrix} \delta_{ij} - \frac{1}{1+k_{d+1,d+1}} k_{d+1,i} k_{d+1,j} & -k_{d+1,i} \\ k_{d+1,j} & k_{d+1,d+1} \end{pmatrix} \\ &= \begin{pmatrix} \delta_{ij} - \frac{2}{1+x^2} x_i x_j & -\frac{2}{1+x^2} x_i \\ \frac{2}{1+x^2} x_j & \frac{1-x^2}{1+x^2} \end{pmatrix} \doteq \tilde{k}_x, \\ x \in \mathbb{R}^d, \quad x_i &= \frac{1}{1+k_{d+1,d+1}} k_{d+1,i}, \quad x^2 = \frac{1-k_{d+1,d+1}}{1+k_{d+1,d+1}}. \end{aligned} \quad (1.32)$$

Note that $k_x \doteq \begin{pmatrix} \tilde{k}_x & 0 \\ 0 & 1 \end{pmatrix}$ appeared in (1.30a) of [183].⁵

Further, we would like to extract from $\hat{C}_\tau$ a representation that may be equivalent to C_χ , $\chi = [\mu, \Delta]$. The first condition for this is that the M representation μ is contained in the restriction of the K representation τ to M . Another condition is that the two

⁵ The matrices k_x realize the (local) isomorphisms: $\mathbb{R}^d \cong \tilde{N} \stackrel{\text{loc}}{\cong} K/M \cong G/MAN$ (using for the last isomorphism the Iwasawa decomposition in the version $G = KAN$).

representations would have the same Casimir values $\lambda_i(\mu, \Delta)$. Having in mind the degeneracy of the Casimir values for partially equivalent representations (see, e. g. (1.25)) we add also the appropriate asymptotic condition. Furthermore, from now on we shall suppose that Δ is real. Thus, we shall use the representations

$$\begin{aligned} \widehat{C}_\chi^\tau &= \{\phi \in \widehat{C}_\tau : C_i(\{\hat{X}\})\phi(x, |a|) = \lambda_i(\mu, \Delta)\phi(x, |a|), \forall i, \\ \mu &\in \tau|_M, \phi(x, |a|) \sim |a|^\Delta \varphi(x) \text{ for } |a| \rightarrow 0\}, \end{aligned} \quad (1.33)$$

where $\{\hat{X}\}$ denotes symbolically the generators of the Lie algebra $\mathcal{G}$ with the action $\hat{X}$ of $X \in \mathcal{G}$ given by the infinitesimal version of (1.29):

$$(\hat{X}\phi)(x, |a|) \doteq \left. \frac{\partial}{\partial t} (\hat{T}^\tau(\exp tX)\phi)(x, |a|) \right|_{t=0}, \quad (1.34)$$

applying the Iwasawa decomposition to $\exp(-tX)\tilde{n}_x a$. Certainly, the Casimir operators $C_i(\{\hat{X}\})$ with (1.34) substituted are differential operators and the elements of $\widehat{C}_\chi^\tau$ are solutions of the equations above.

Remark 3. Note that generically the functions in (1.33) have also a second limit with $\Delta \rightarrow d - \Delta$:

$$\tilde{\varphi}(x) = \lim_{|a| \rightarrow 0} |a|^{\Delta-d} \phi(x, |a|), \quad (1.35)$$

which will appear as a consequence of the formalism. With this we shall establish—for generic representations—the following important relation:

$$\widehat{C}_\chi^\tau = \widehat{C}_{\tilde{\chi}}^\tau, \quad \chi = [\mu, \Delta], \quad \tilde{\chi} = [\tilde{\mu}, d - \Delta], \quad (1.36)$$

for which, besides (1.35), we use the equality between the Casimir operators (1.25), and the fact that if τ contains μ then it also contains the mirror image $\tilde{\mu}$. This is established towards the end of the next section, where also a comment on the exceptional cases is made. $\diamond$

We end this section by noting that for the representations on de Sitter space $\widehat{C}_\chi, \widehat{C}_\tau$, there is no exhaustivity result as the Langlands–Knapp–Zuckerman result [405, 391] for ERs cited above. Thus, it is not surprising that not all conformal representations can be realized on de Sitter space, or, in other words, that some conformal fields live only on the boundary of de Sitter space and cannot propagate into the bulk.

1.1.5 Intertwining relations between conformal and de Sitter representations

Bulk-to-boundary intertwining relation

This section contains our main results, explicating the relations between CFT and de Sitter representations as intertwining relations. We first give in this subsection the

intertwining operator from the de Sitter to the CFT realization. The operator which we use maps a function on de Sitter space to its boundary value and was used in a restricted sense (explained below) in many papers, starting from [572]. Also for those cases our result is new since we use it as operator between exactly defined spaces, and most importantly we give it the interpretation of an intertwining operator.

Theorem 1. *Let us define the operator:*

$$L_\chi^\tau : \widehat{C}_\chi^\tau \longrightarrow C_\chi, \quad (1.37)$$

with the following action:

$$(L_\chi^\tau \phi)(x) = \lim_{|a| \rightarrow 0} |a|^{-\Delta} \Pi_\mu^\tau \phi(x, |a|) \quad (1.38)$$

where Π_μ^τ is the standard projection operator from the K -representation space U_τ to the M -representation space V_μ , which acts in the following way on the K -representation matrices:

$$\Pi_\mu^\tau \tilde{D}^\tau(k) = \hat{d}^\mu(m(k)) \Pi_\mu^\tau \tilde{D}^\tau(k_f) \quad (1.39)$$

where we have used (1.31). Then L_χ^τ is an intertwining operator, i. e.

$$L_\chi^\tau \circ \hat{T}^\tau(g) = T^\chi(g) \circ L_\chi^\tau, \quad \forall g \in G. \quad (1.40)$$

In addition, in (1.38) the operator Π_μ^τ acts in the following truncated way:

$$\Pi_\mu^\tau \tilde{D}^\tau(k) = \hat{d}^\mu(m(k)) \Pi_\mu^\tau. \quad (1.41)$$

Proof. Applying the LHS side of (1.40) to ϕ we have

$$\begin{aligned} (L_\chi^\tau \circ \hat{T}^\tau(g)\phi)(x) &= \lim_{|a| \rightarrow 0} |a|^{-\Delta} \Pi_\mu^\tau (\hat{T}^\tau(g)\phi)(x, |a|) \\ &= \lim_{|a| \rightarrow 0} |a|^{-\Delta} \hat{d}^\mu(m(k)) \Pi_\mu^\tau \tilde{D}^\tau(k_f) \phi(x', |a'|), \end{aligned} \quad (1.42a)$$

$$g^{-1} \tilde{n}_x a = \tilde{n}_{x'} a' k^{-1}. \quad (1.42b)$$

Applying the RHS side of (1.40) to ϕ we have

$$\begin{aligned} (T^\chi(g) \circ L_\chi^\tau \phi)(x) &= |a''|^{-\Delta} \hat{d}^\mu(m) (L_\chi^\tau \phi)(x'') \\ &= |a''|^{-\Delta} \hat{d}^\mu(m) \lim_{|a| \rightarrow 0} |a|^{-\Delta} \Pi_\mu^\tau \phi(x'', |a|), \end{aligned} \quad (1.43a)$$

$$g^{-1} \tilde{n}_x = \tilde{n}_{x''} m^{-1} a''^{-1} n^{-1}. \quad (1.43b)$$

In view of the Bruhat decomposition it is enough to prove coincidence between (1.42a) and (1.43a) for $g = \tilde{n}_y \in \tilde{N}$, $\hat{a} \in A$, $\hat{m} \in M$, and some element $w \in K$, $w \notin M$, representing the nontrivial element of the restricted Weyl group $W(G, A)$, since this element transforms elements of $\tilde{N}$ into elements of N : $w \tilde{n}_y w = n_{y'}$ (and thus makes it

unnecessary to check $g = n_y$). Such an element is, e. g. $w = \text{diag}(1, \dots, 1, -1, -1, 1)$, i. e. rotation in the plane $(d, d+1)$. However, for simplicity we shall demonstrate only the case of odd d when we can take $w \rightarrow R$, since in this case the conformal inversion $R \doteq \text{diag}(-1, \dots, -1, 1)$ is an element of K (or we should suppose that we work with G'). We have the following cases.

- $g = \tilde{n}_y$: then (1.42b) gives $\tilde{n}_y^{-1} \tilde{n}_x a = \tilde{n}_{x-y} a$, i. e., $x' = x - y$, $a' = a$, $k = \mathbf{1}$, and (noting $\tilde{D}^\tau(\mathbf{1}) = \mathbf{1}_\tau$) (1.42a) becomes

$$\lim_{|a| \rightarrow 0} |a|^{-\Delta} \Pi_\mu^\tau \phi(x - y, |a|), \quad (1.44)$$

while (1.43b) gives $\tilde{n}_y^{-1} \tilde{n}_x = \tilde{n}_{x-y}$, i. e., $x'' = x - y$, $m = \mathbf{1}$, $a'' = \mathbf{1}$, $n = \mathbf{1}$, and (noting $\hat{d}^\mu(\mathbf{1}) = \mathbf{1}_\mu$) (1.43a) also becomes (1.44).

- $g = \hat{a}$: then (1.42b) gives $\hat{a}^{-1} \tilde{n}_x a = \tilde{n}_{\frac{x}{|\hat{a}|}} \hat{a}^{-1} a$, i. e., $x' = \frac{x}{|\hat{a}|}$, $a' = \hat{a}^{-1} a$, $k = \mathbf{1}$, and (1.42a) becomes

$$\lim_{|a| \rightarrow 0} |a|^{-\Delta} \Pi_\mu^\tau \phi\left(\frac{x}{|\hat{a}|}, \frac{|a|}{|\hat{a}|}\right) = |\hat{a}|^{-\Delta} \lim_{|a| \rightarrow 0} |a|^{-\Delta} \Pi_\mu^\tau \phi\left(\frac{x}{|\hat{a}|}, |a|\right), \quad (1.45)$$

while (1.43b) gives $\hat{a}^{-1} \tilde{n}_x = \tilde{n}_{\frac{x}{|\hat{a}|}} \hat{a}^{-1}$, i. e., $x'' = \frac{x}{|\hat{a}|}$, $m = \mathbf{1}$, $a'' = \hat{a}$, $n = \mathbf{1}$, and (1.43a) also becomes (1.45).

- $g = \hat{m}$: then (1.42b) gives $\hat{m}^{-1} \tilde{n}_x a = \tilde{n}_{\hat{m}^{-1}x} \hat{m}^{-1} a$, i. e., $x'_i = m_{ij}^{-1} x_j$, $a' = a$, $k = m(k) = \hat{m}$, and (1.42a) becomes

$$\lim_{|a| \rightarrow 0} |a|^{-\Delta} \Pi_\mu^\tau \tilde{D}^\tau(\hat{m}) \phi(x', |a|) = \hat{d}^\mu(\hat{m}) \lim_{|a| \rightarrow 0} |a|^{-\Delta} \Pi_\mu^\tau \phi(x', |a|), \quad (1.46)$$

while (1.43b) gives $\hat{m}^{-1} \tilde{n}_x = \tilde{n}_{\hat{m}^{-1}x} \hat{m}^{-1}$, i. e., $x'' = x'$, $m = \hat{m}$, $a'' = \mathbf{1}$, $n = \mathbf{1}$, and (1.43a) also becomes (1.46).

- $g = R$: then (1.42b) gives $R \tilde{n}_x a = \tilde{n}_{x'} a' k^{-1}$ with

$$\begin{aligned} x' &= x(x, a) \equiv -\frac{1}{x^2 + |a|^2} x, \\ a' &= a(x, a)^{-1} a, \quad |a(x, a)| = x^2 + |a|^2, \\ k &= k(x, a) \equiv \begin{pmatrix} \frac{2}{x^2 + |a|^2} x_i x_j - \delta_{ij} & -\frac{2|a|}{x^2 + |a|^2} x_i & 0 \\ \frac{2|a|}{x^2 + |a|^2} x_j & \frac{x^2 - |a|^2}{x^2 + |a|^2} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned} \quad (1.47)$$

Using (1.32) we note for later use that $k(x, \mathbf{1}) = -{}^t k_x$, and also

$$k(x, a) = m(k(x, a)) k(x, |a|)_f = r(x) k_{\frac{|a|}{x}}. \quad (1.48)$$

Here we first record (for $x \neq 0$)

$$\lim_{|a| \rightarrow 0} x(x, a) = -\frac{1}{x^2} x \equiv R x,$$

$$\begin{aligned}\lim_{|a| \rightarrow 0} a(x, a) &= a(x), \quad |a(x)| = x^2, \\ \lim_{|a| \rightarrow 0} k(x, a) &= r(x),\end{aligned}\tag{1.49}$$

and (1.42a) becomes

$$\begin{aligned}\lim_{|a| \rightarrow 0} |a|^{-\Delta} \Pi_\mu^\tau \tilde{D}^\tau(k(x, a)) \phi(x(x, a), |a(x, a)|^{-1} |a|) \\ = (x^2)^{-\Delta} \hat{d}^\mu(r(x)) \lim_{|a| \rightarrow 0} |a|^{-\Delta} \Pi_\mu^\tau \phi(Rx, |a|).\end{aligned}\tag{1.50}$$

On the other hand (1.43b) gives (also for $x \neq 0$): $R\tilde{n}_x = \tilde{n}_{x''} m^{-1} a''^{-1} n^{-1}$ with $x'' = Rx$, $m = r(x)$, $a'' = a(x)$, using the notation introduced in (1.49). Thus, (1.43a) also becomes (1.50).

This finishes the proof of the intertwining property. On the way we have proved also (1.41). Indeed, though we have started with (1.39) in the generic formula (1.42a) for the LHS of the intertwining property, in the four generating cases above we have $k_f = \mathbf{1}$, i. e., we could have started with (1.41) in (1.42a). $\square$

Boundary-to-bulk intertwining relation

Now we look for the possible operator inverse to L_χ^τ which would restore a function on de Sitter space from its boundary value, as discussed in [572, 136]. Again what is new here is that we define it as intertwining operator between exactly defined spaces in a more general setting. Moreover, we shall construct the operator just from the condition that it is an intertwining integral operator. Indeed, let us take the operator

$$\tilde{L}_\chi^\tau : C_\chi \longrightarrow \hat{C}_\chi^\tau,\tag{1.51}$$

and try the following Ansatz:

$$(\tilde{L}_\chi^\tau f)(x, |a|) = \int K_\chi^\tau(x, |a|; x') f(x') dx' \tag{1.52}$$

where $K_\chi^\tau(x, |a|; x')$ is a linear operator acting from the space V_μ to the space U_τ , and let us suppose that $\tilde{L}_\chi^\tau$ is an intertwining operator, i. e.,

$$\hat{T}^\tau(g) \circ \tilde{L}_\chi^\tau = \tilde{L}_\chi^\tau \circ T^\chi(g), \quad \forall g \in G.\tag{1.53}$$

As in Theorem 1, we apply (1.53) for $g = \tilde{n}_y, \hat{a}, \hat{m}, R$ and we use the same decompositions as above, so we can present things in a short fashion. Applying (1.53) for $g = \tilde{n}_y$ results in the fact that K_χ^τ depends only on the difference of the x arguments:

$$K_\chi^\tau(x, |a|; x') = K_\chi^\tau(x - x', |a|).\tag{1.54}$$

Applying (1.53) for $g = \hat{a}$ results in the fact that K_χ^τ is homogeneous in its arguments:

$$K_\chi^\tau(\sigma x, \sigma |a|) = \sigma^{\Delta-d} K_\chi^\tau(x, |a|), \quad \sigma \in \mathbb{C}, \sigma \neq 0. \quad (1.55)$$

Thus, we shall write

$$K_\chi^\tau(x, |a|) = |a|^{\Delta-d} \hat{K}_\chi^\tau\left(\frac{x}{|a|}\right), \quad \hat{K}_\chi^\tau(y) \doteq K_\chi^\tau(y, 1). \quad (1.56)$$

Note now that (1.56) means, in particular, that $K_\chi^\tau(0, |a|)$ (if it exists) is fixed up to a constant matrix:

$$K_\chi^\tau(0, |a|) = |a|^{\Delta-d} K_\chi^\tau(0, 1) = |a|^{\Delta-d} \hat{K}_\chi^\tau(0). \quad (1.57)$$

Applying (1.53) for $g = \hat{m}$ results in the fact that K_χ^τ has the following covariance property:

$$\tilde{D}^\tau(m) \hat{K}_\chi^\tau(x) = \hat{K}_\chi^\tau(mx) \hat{d}^\mu(m), \quad \forall m \in M \quad (1.58)$$

The above means, in particular, that $\Pi_\mu^\tau \hat{K}_\chi^\tau(0)$ is M -invariant:

$$\hat{d}^\mu(m) \Pi_\mu^\tau \hat{K}_\chi^\tau(0) = \Pi_\mu^\tau \hat{K}_\chi^\tau(0) \hat{d}^\mu(m), \quad \forall m \in M, \quad (1.59)$$

which then, by Schur's lemma, means that $\Pi_\mu^\tau \hat{K}_\chi^\tau(0) = \sigma \mathbf{1}_\mu$, $0 \neq \sigma \in \mathbb{C}$. Thus, we have $\hat{K}_\chi^\tau(0) = \sigma \Pi_\tau^\mu$, the latter being the canonical embedding operator from V_μ to U_τ , such that $\Pi_\mu^\tau \circ \Pi_\tau^\mu = \mathbf{1}_\mu$. Finally, applying (1.53) for $g = R$ means that

$$\begin{aligned} \tilde{D}^\tau(r(x)) \tilde{D}^\tau(k_{\frac{|a|}{x^2}x}) \hat{K}_\chi^\tau\left(-\frac{x}{|a|} + \frac{x^2 + |a|^2}{|a|y^2}y\right) \\ = \left(\frac{y^2}{x^2 + |a|^2}\right)^{d-\Delta} \hat{K}_\chi^\tau\left(\frac{x-y}{|a|}\right) \hat{d}^\mu(r(y)) \end{aligned} \quad (1.60)$$

where we have used the decomposition of $k(x, |a|)$ in (1.48). Now we set $y = x$ and we get

$$\tilde{D}^\tau(r(x)) \tilde{D}^\tau(k_{\frac{|a|}{x^2}x}) \hat{K}_\chi^\tau\left(\frac{|a|}{x^2}x\right) = \left(\frac{x^2}{x^2 + |a|^2}\right)^{d-\Delta} \hat{K}_\chi^\tau(0) \hat{d}^\mu(r(x)). \quad (1.61)$$

Using (1.58) for $x = 0$ we obtain (using also that $r(x)^2 = \mathbf{1}$)

$$\tilde{D}^\tau(k_{\frac{|a|}{x^2}x}) \hat{K}_\chi^\tau\left(\frac{|a|}{x^2}x\right) = \left(\frac{x^2}{x^2 + |a|^2}\right)^{d-\Delta} \hat{K}_\chi^\tau(0). \quad (1.62)$$

Next, we make the change $\frac{|a|}{x^2}x \rightarrow x$ and we get

$$\hat{K}_\chi^\tau(x) = N_\chi^\tau \left(\frac{1}{x^2 + 1}\right)^{d-\Delta} \tilde{D}^\tau(k_{-x}) \Pi_\tau^\mu \quad (1.63)$$

or finally

$$K_\chi^\tau(x, |a|) = N_\chi^\tau \left(\frac{|a|}{x^2 + |a|^2} \right)^{d-\Delta} \tilde{D}^\tau(k_{-\frac{x}{|a|}}) \Pi_\tau^\mu \quad (1.64)$$

where N_χ^τ is arbitrary for the moment and should be fixed from the requirement that $\tilde{L}_\chi^\tau$ is inverse to L_χ^τ (when the latter is true).

The above operator exists for arbitrary representations τ of $K = \text{SO}(d+1)$ which contain the representation μ of $M = \text{SO}(d)$. The condition that $\tau = [\ell'_1, \dots, \ell'_d]$ ($\hat{d} \equiv [\frac{d+1}{2}]$) contains $\mu = [\ell_1, \dots, \ell_d]$ ($\tilde{d} \equiv [\frac{d}{2}]$) explicitly is (cf. (1.11), (1.12), and (1.13))

$$|\ell'_1| \leq \ell_1 \leq \dots \leq \ell_{\tilde{d}} \leq \ell'_d, \quad d \text{ odd}, \quad \hat{d} = \tilde{d} + 1, \quad (1.65a)$$

$$|\ell'_1| \leq \ell'_1 \leq \dots \leq \ell_{\tilde{d}} \leq \ell'_d, \quad d \text{ even}, \quad \hat{d} = \tilde{d}. \quad (1.65b)$$

If one is primarily concerned with the ERs $\chi = [\mu, \Delta]$ it is convenient to choose a ‘minimal’ representation $\tau(\mu)$ of $K = \text{SO}(d+1)$ containing μ . This depends on the parity of d . Thus, for μ as above, when d is *odd* we would choose

$$\tau(\mu) = [\ell_1, \ell_2, \dots, \ell_d] \quad \text{or} \quad \tilde{\tau}(\mu) = [-\ell_1, \ell_2, \dots, \ell_d], \quad \mu \cong \tilde{\mu}, \quad (1.66)$$

while for *even* d we would choose

$$\tau(\mu) = [|\ell_1|, \ell_2, \dots, \ell_d] = \tau(\tilde{\mu}) \cong \tilde{\tau}(\mu) = \tilde{\tau}(\tilde{\mu}) \quad (1.67)$$

Thus, in the odd d case for each μ we would choose between two K -irreps which are mirror images of one another, while in the even d case to each two mirror-image irreps of M we choose one and the same irrep of K .

The explicit formulae which appeared until now in the literature are actually in the cases in which $\tau = \tau(\mu)$, though there is no such interpretation as we have here. In such a restricted setting and from other considerations formula (1.64) for the scalar case (when both μ and $\tau = \tau(\mu)$ are scalar irreps) was given by Witten [572], while some other nonscalar cases were given in [572, 426, 458]. Note that in (2.38) of [572] it is written for the conjugated conformal weight: $\Delta \rightarrow d - \Delta$, which in our language would mean to work with the representation $\tilde{\chi} = [\tilde{\mu}, d - \Delta]$ and to use

$$K_{\tilde{\chi}}^\tau(x, |a|) = N_{\tilde{\chi}}^\tau \left(\frac{|a|}{x^2 + |a|^2} \right)^\Delta \tilde{D}^\tau(k_{-\frac{x}{|a|}}) \Pi_{\tilde{\tau}}^{\tilde{\mu}}. \quad (1.68)$$

Equivalence vs. partial equivalence

Either one of the representation equivalences established in the previous subsections means that the representations $\hat{C}_\chi^\tau$ and C_χ are *partially equivalent representations*. In

order for them to be *equivalent representations* it is necessary and sufficient that the operators $\tilde{L}_\chi^\tau, L_\chi^\tau$ are inverse to each other, i. e., the following relations should hold:

$$L_\chi^\tau \circ \tilde{L}_\chi^\tau = \mathbf{1}_{C_\chi}, \quad \tilde{L}_\chi^\tau \circ L_\chi^\tau = \mathbf{1}_{\tilde{C}_\chi^\tau} \quad (1.69)$$

If (1.69) does not hold then at least one of the representations $\tilde{C}_\chi^\tau$ and C_χ^τ is topologically reducible.

For the first relation in (1.69) we have

$$\begin{aligned} (L_\chi^\tau \circ \tilde{L}_\chi^\tau f)(x) &= \lim_{|a| \rightarrow 0} |a|^{-\Delta} \Pi_\mu^\tau(\tilde{L}_\chi^\tau f)(x, |a|) \\ &= \lim_{|a| \rightarrow 0} |a|^{-\Delta} \Pi_\mu^\tau \int K_\chi^\tau(x - x', |a|) f(x') dx'. \end{aligned} \quad (1.70)$$

For the above calculation we interchange the limit and the integration, and use the following result from [572] (there for $\Delta > d/2$):

$$\lim_{|a| \rightarrow 0} |a|^{\Delta-d} \left(\frac{|a|}{x^2 + |a|^2} \right)^\Delta \sim \delta(x) \quad (1.71)$$

substituting for our needs $\Delta \rightarrow d - \Delta$.

To obtain the proportionality constant in (1.71), and thus fix N_χ^τ , we first find the Fourier transform of $(x^2 + |a|^2)^{-\Delta}$:

$$\begin{aligned} \int \frac{e^{-ip \cdot x}}{(x^2 + |a|^2)^\Delta} \frac{dx}{(2\pi)^{d/2}} &= \int_0^\infty \frac{dx d\alpha \alpha^{\Delta-1} e^{-ip \cdot x - \frac{1}{2}\alpha(x^2 + |a|^2)}}{(2\pi)^{d/2} 2^\Delta \Gamma(\Delta)} \\ &= \frac{1}{2^\Delta \Gamma(\Delta)} \int_0^\infty d\alpha \alpha^{\Delta - \frac{d}{2} - 1} e^{-\frac{1}{2}(\frac{p^2}{\alpha} + \alpha|a|^2)} \\ &= \frac{2}{2^\Delta \Gamma(\Delta)} \left(\frac{\sqrt{p^2}}{|a|} \right)^{\Delta - \frac{d}{2}} K_{\Delta - \frac{d}{2}}(|a| \sqrt{p^2}) \end{aligned} \quad (1.72)$$

where we have used formula (3.47.9) of [287], involving the Bessel function K_ν . Note that for $|a| \rightarrow 0$ the above formula goes over to formula (5.2) of [183] (with $\Delta = h + c$), in particular, the RHS of (1.72) goes over to

$$\frac{\Gamma(\frac{d}{2} - \Delta)}{2^\Delta \Gamma(\Delta)} \left(\frac{p^2}{2} \right)^{\Delta - \frac{d}{2}}, \quad \frac{d}{2} - \Delta \notin \mathbb{Z}_-. \quad (1.73)$$

For (1.73) one uses the relation

$$\lim_{\beta \rightarrow 0} 2\beta^\nu K_\nu(\beta) = 2^\nu \Gamma(\nu), \quad \nu \notin \mathbb{Z}_-. \quad (1.74)$$

Now we can find the necessary limit in (1.71) (again using (1.74)):

$$\begin{aligned}
& \lim_{|a| \rightarrow 0} |a|^{\Delta-d} \left(\frac{|a|}{x^2 + |a|^2} \right)^\Delta \\
&= \lim_{|a| \rightarrow 0} \frac{2|a|^{2\Delta-d}}{2^\Delta \Gamma(\Delta)} \int \left(\frac{\sqrt{p^2}}{|a|} \right)^{\Delta-\frac{d}{2}} K_{\Delta-\frac{d}{2}}(|a|\sqrt{p^2}) \frac{e^{ip \cdot x} dp}{(2\pi)^{d/2}} \\
&= \lim_{|a| \rightarrow 0} \frac{2}{2^\Delta \Gamma(\Delta)} \int (|a|\sqrt{p^2})^{\Delta-\frac{d}{2}} K_{\Delta-\frac{d}{2}}(|a|\sqrt{p^2}) \frac{e^{ip \cdot x} dp}{(2\pi)^{d/2}} \\
&= \frac{\Gamma(\Delta - \frac{d}{2})}{2^{\frac{d}{2}} \Gamma(\Delta)} \int \frac{e^{ip \cdot x} dp}{(2\pi)^{d/2}} = \frac{\pi^{\frac{d}{2}} \Gamma(\Delta - \frac{d}{2})}{\Gamma(\Delta)} \delta(x)
\end{aligned} \tag{1.75}$$

where we have used

$$\delta(x) = \int \frac{e^{ip \cdot x} dp}{(2\pi)^d}. \tag{1.76}$$

In the scalar case prompted by (1.75) (with $\Delta \rightarrow d - \Delta$) we choose the constant in (1.64) as

$$N_\chi^\tau = \frac{\Gamma(d - \Delta)}{\pi^{\frac{d}{2}} \Gamma(\frac{d}{2} - \Delta)}, \quad \Delta \neq d + k, \quad k \in \mathbb{Z}_+ \text{ for } d \text{ odd}. \tag{1.77}$$

This is the choice made in [242] (for $\Delta \rightarrow d - \Delta$) from other considerations. In the general case we choose N_χ^τ as

$$\begin{aligned}
N_\chi^\tau &= N_0 \frac{\Gamma(\hat{d} - \Delta)}{\Gamma(\frac{\hat{d}}{2} - \Delta)} \prod_{k=1}^{\tilde{d}} \left(m_k + \frac{d}{2} - \Delta \right), \\
m_k &\doteq |\ell_k + k - 1 + \delta_d|, \quad k = 1, \dots, \tilde{d}, \\
\tilde{d} &\equiv \left\lfloor \frac{d}{2} \right\rfloor, \quad \hat{d} \equiv \left\lfloor \frac{d+1}{2} \right\rfloor, \quad \delta_d = \frac{d}{2} - \tilde{d} = \begin{cases} 0 & d \text{ even,} \\ \frac{1}{2} & d \text{ odd,} \end{cases}
\end{aligned} \tag{1.78}$$

where N_0 is a constant independent of Δ having no poles or zeroes for any χ . Note that for d odd N_χ^τ has poles and thus is not defined in the following cases:

$$\Delta = \frac{d+1}{2} + p, \quad p \in \mathbb{Z}_+, \quad p \neq m_k - \frac{1}{2}, \quad k = 1, \dots, \tilde{d}, \quad d \text{ odd}, \tag{1.79}$$

(which for the scalar case coincide with the exclusion conditions in (1.77)). We note also the zero cases:

$$N_\chi^\tau = 0, \quad \text{for } \begin{cases} \Delta = m_k + \frac{d}{2}, \quad k = 1, \dots, \tilde{d}, & d \text{ even} \\ \Delta = m_k + \frac{d}{2} \neq \frac{d+1}{2} + p, \quad k = 1, \dots, \tilde{d}, \quad p \in \mathbb{Z}_+, & d \text{ odd} \\ \Delta = \frac{d}{2} + p, \quad p \in \mathbb{Z}_+. & d \text{ odd} \end{cases} \tag{1.80}$$

Further we need also the following fact, which for simplicity we write for d odd (or for $G \rightarrow G'$):

$$\lim_{|a| \rightarrow 0} k_{-\frac{x}{|a|}} = r(x)R. \quad (1.81)$$

With the choice (1.77) (or (1.78)) and using (1.75) and (1.81) the last line of (1.70) gives $f(x)$, thus establishing the first relation in (1.69) for generic values of Δ (i. e., when N_χ^τ is finite and nonzero).

As a corollary we recover the fact [572] that for generic values of Δ we can restore a function on de Sitter space from its boundary value on R^d . Indeed, suppose we have

$$\phi(x, |a|) = \int K_\chi^\tau(x - x', |a|) f(x') dx', \quad \phi \in \hat{C}_\chi^\tau, f \in C_\chi \quad (1.82)$$

Then we have for the boundary value

$$\begin{aligned} \psi_0(x) &\doteq (L_\chi^\tau \phi)(x) = \lim_{|a| \rightarrow 0} |a|^{-\Delta} \Pi_\mu^\tau \phi(x, |a|) \\ &= \lim_{|a| \rightarrow 0} |a|^{-\Delta} \Pi_\mu^\tau \int K_\chi^\tau(x - x', |a|) f(x') dx' = f(x). \end{aligned} \quad (1.83)$$

Now we can prove the second relation in (1.69):

$$\begin{aligned} (\tilde{L}_\chi^\tau \circ L_\chi^\tau \phi)(x, |a|) &= \int K_\chi^\tau(x - x', |a|) (L_\chi^\tau \phi)(x') dx' \\ &= \int K_\chi^\tau(x - x', |a|) \lim_{\tilde{v}_0 \rightarrow 0} \tilde{v}_0^{-\Delta} \Pi_\mu^\tau \phi(x', \tilde{v}_0) \\ &= \int K_\chi^\tau(x - x', |a|) \psi_0(x') dx' = \phi(x, |a|) \end{aligned} \quad (1.84)$$

where in the last line we used (1.83).

• Thus, we have found that the partially equivalent representations $\hat{C}_\chi^\tau$ and C_χ ($\chi = [\mu, \Delta]$) are equivalent iff Δ is not in the excluded ranges given in (1.79), (1.80).

This result may be used for the conjugate situation $\chi \rightarrow \bar{\chi}$. The constant then is

$$N_\chi^\tau = \tilde{N}_0 \frac{\Gamma(\Delta - \tilde{d})}{\Gamma(\Delta - \frac{d}{2})} \prod_{k=1}^{\tilde{d}} \left(m_k - \frac{d}{2} + \Delta \right) \quad (1.85)$$

where $\tilde{N}_0$ is a constant independent of Δ having no poles or zeroes for any χ . The cases when N_χ^τ is not defined are

$$\Delta = \tilde{d} - p, \quad p \in \mathbb{Z}_+, \quad p \neq m_k - \frac{1}{2}, \quad k = 1, \dots, \tilde{d}, \quad d \text{ odd}, \quad (1.86)$$

while the zero cases are

$$N_\chi^\tau = 0, \quad \text{for} \quad \begin{cases} \Delta = \frac{d}{2} - m_k, \quad k = 1, \dots, \tilde{d}, & d \text{ even} \\ \Delta = \frac{d}{2} - m_k \neq \frac{d+1}{2} + p, \quad k = 1, \dots, \tilde{d}, \quad p \in \mathbb{Z}_+, & d \text{ odd} \\ \Delta = \frac{d}{2} - p, \quad p \in \mathbb{Z}_+. & d \text{ odd} \end{cases} \quad (1.87)$$

- Thus, we find that the partially equivalent representations $\hat{C}_{\tilde{\chi}}^\tau$ and $C_{\tilde{\chi}}$ ($\tilde{\chi} = [\tilde{\mu}, d - \Delta]$) are equivalent iff Δ is not in the excluded ranges given in (1.86), (1.87).

Further intertwining relations

We start by recording the second limit of the bulk functions (which we mentioned towards the end of Section 1.1.4). We take as before $\psi_0 \in C_\chi$, $\chi = [\mu, \Delta]$, and $\phi \in \hat{C}_\chi^\tau$ expressed through ψ_0 by (1.82) and (1.83). We set and calculate

$$\begin{aligned}
 \phi_0(x) &\doteq \lim_{|a| \rightarrow 0} |a|^{\Delta-d} \Pi_\mu^\tau \phi(x, |a|) \\
 &= \lim_{|a| \rightarrow 0} |a|^{\Delta-d} \Pi_\mu^\tau \int K_\chi^\tau(x - x', |a|) \psi_0(x') dx' \\
 &= N_\chi^\tau \int \frac{dx'}{(x - x')^{2(d-\Delta)}} \Pi_\mu^\tau \tilde{D}^\tau(r(x)R) \Pi_\tau^\mu \psi_0(x') \\
 &= N_\chi^\tau \int \frac{dx'}{(x - x')^{2(d-\Delta)}} \hat{d}^\mu(r(x)) \psi_0(x') \\
 &= \frac{N_\chi^\tau}{\gamma_{\tilde{\chi}}} \int dx' G_{\tilde{\chi}}(x - x') \psi_0(x'). \tag{1.88}
 \end{aligned}$$

Since $\psi_0 \in C_\chi$ from the intertwining property of $G_{\tilde{\chi}}$ it follows that $\phi_0 \in C_{\tilde{\chi}}$, $\tilde{\chi} = [\tilde{\mu}, d - \Delta]$. This is valid for generic representations—in the exceptional cases it may happen that $\phi_0 = 0$ (when ψ_0 is in the kernel of $G_{\tilde{\chi}}$) or that the asymptotic expansion contains logarithms, (for the latter, cf., e. g. (7.45) of [183]).

Thus, we have established (1.36) at generic representation points. The two different limits of the bulk field ϕ are given by the coupled fields ϕ_0 and ψ_0 (the latter we denote also by $\mathcal{O}$).

Formulae (1.88) mean that if we define

$$\mathcal{A}_\chi^\tau : \hat{C}_\chi^\tau \longrightarrow C_{\tilde{\chi}} \tag{1.89}$$

with the action

$$(\mathcal{A}_\chi^\tau \phi)(x) = \lim_{|a| \rightarrow 0} |a|^{\Delta-d} \Pi_\mu^\tau \phi(x, |a|) \tag{1.90}$$

then we can show as in the theorem the intertwining property:

$$\mathcal{A}_{\tilde{\chi}}^\tau \circ \hat{T}^\tau(g) = T^{\tilde{\chi}}(g) \circ \mathcal{A}_\chi^\tau, \quad \forall g \in G, \tag{1.91}$$

since in the proof of the theorem we have used only the fact of the existence of the limit. On the other hand $\mathcal{A}_\chi^\tau$ is equal to L_χ^τ because of (1.36).

Next we note that the intertwining property (1.91) is fulfilled if we take the following as a defining relation:

$$\mathcal{A}_{\tilde{\chi}}'^\tau = G_{\tilde{\chi}} \circ L_\chi^\tau. \tag{1.92}$$

Expectedly, we get the same result as in (1.88) (up to a multiplicative constant):

$$\begin{aligned}
 (\mathcal{A}'_{\tilde{\chi}}\phi)(x) &= (G_{\tilde{\chi}} \circ L_{\tilde{\chi}}^{\tau}\phi)(x) \\
 &= \int dx' G_{\tilde{\chi}}(x - x')(L_{\tilde{\chi}}^{\tau}\phi)(x') \\
 &= \int dx' G_{\tilde{\chi}}(x - x')\psi_0(x').
 \end{aligned} \tag{1.93}$$

Gathering everything we have in this subsection we obtain the following relation between three of the operators under consideration:

$$L_{\tilde{\chi}}^{\tau} = \frac{N_{\tilde{\chi}}^{\tau}}{\gamma_{\tilde{\chi}}} G_{\tilde{\chi}} \circ L_{\tilde{\chi}}^{\tau}. \tag{1.94}$$

At generic points from this we can obtain a lot of interesting relations, e. g. applying $\tilde{L}_{\chi}^{\tau}$ from the right we get

$$L_{\tilde{\chi}}^{\tau} \circ \tilde{L}_{\chi}^{\tau} = \frac{N_{\tilde{\chi}}^{\tau}}{\gamma_{\tilde{\chi}}} G_{\tilde{\chi}}, \tag{1.95}$$

which is in fact (1.88). Applying G_{χ} from the left we get

$$L_{\chi}^{\tau} = \frac{\gamma_{\tilde{\chi}}}{N_{\chi}^{\tau}} G_{\chi} \circ L_{\tilde{\chi}}^{\tau} \tag{1.96}$$

and making the change $\chi \longleftrightarrow \tilde{\chi}$ in the last equality we get

$$L_{\tilde{\chi}}^{\tau} = \frac{\gamma_{\chi}}{N_{\tilde{\chi}}^{\tau}} G_{\tilde{\chi}} \circ L_{\chi}^{\tau} \tag{1.97}$$

Comparing (1.94) and (1.97), we obtain the following relation between the normalization constants:

$$N_{\chi}^{\tau} N_{\tilde{\chi}}^{\tau} = \gamma_{\chi} \gamma_{\tilde{\chi}} = C\rho(\chi) \tag{1.98}$$

where $\rho(\chi)$ is the analytic continuation of the Plancherel measure for the Plancherel formula contribution of the principal series of unitary irreps of G , and the last equality was shown in [390]. (The constant C is independent of χ .) The Plancherel measure itself is given as follows [463, 317]:⁶

$$\begin{aligned}
 \rho(\chi) &= \rho(\tilde{\chi}) = \rho_0(\mu) \left(\prod_{1 \leq i < j \leq \tilde{d}} (m_j^2 - m_i^2) \right) \\
 &\times \frac{\Gamma(\delta_d + c) \Gamma(\delta_d - c)}{\Gamma(c) \Gamma(-c)} \prod_{k=1}^{\tilde{d}} (m_k^2 - c^2),
 \end{aligned} \tag{1.99a}$$

⁶ The principal series is obtained for c purely imaginary.

$$c \equiv \Delta - \frac{d}{2}, \quad \rho_0(\mu) = \begin{cases} 1 & d \text{ even,} \\ \prod_{k=1}^{\tilde{d}} m_k & d \text{ odd;} \end{cases} \quad (1.99b)$$

cf. (1.78). In the scalar case, when $\ell_j = 0$ for all j , we have

$$\rho(\chi) = C' \frac{\Gamma(\frac{d}{2} + c)\Gamma(\frac{d}{2} - c)}{\Gamma(c)\Gamma(-c)} = C' \frac{\Gamma(\Delta)\Gamma(d - \Delta)}{\Gamma(\Delta - \frac{d}{2})\Gamma(\frac{d}{2} - \Delta)} \quad (1.100)$$

the constant C' being independent of Δ .

Another consequence of (1.98) is that the constants N_χ^τ do not depend on τ since the constants γ_χ do not. Thus, we drop their superscript τ .

The above relation between these constants may be obtained also if we use the reconstruction of the field ϕ from ϕ_0 :

$$\phi(x, |a|) = \int K_\chi^\tau(x - x', |a|) \phi_0(x') dx' = (\tilde{L}_\chi^\tau \phi_0)(x, |a|). \quad (1.101)$$

Proceeding as in (1.88) we have

$$\begin{aligned} \psi_0(x) &= (L_\chi^\tau \phi)(x) = \frac{N_{\tilde{\chi}}}{\gamma_\chi} \int dx' G_\chi(x - x') \phi_0(x') \\ &= \frac{N_\chi N_{\tilde{\chi}}}{\gamma_\chi \gamma_{\tilde{\chi}}} \int dx' dx'' G_\chi(x - x') G_{\tilde{\chi}}(x' - x'') \psi_0(x'') \\ &= \frac{N_\chi N_{\tilde{\chi}}}{\gamma_\chi \gamma_{\tilde{\chi}}} \psi_0(x) \end{aligned} \quad (1.102)$$

where we have substituted (1.88) in the last but one line, and used (1.26) in the last line.

Another interesting relation includes a convolution of a K -kernel and a G -kernel. For this purpose, we apply $\tilde{L}_\chi^\tau$ to (1.95) from the left and changing $\chi \longleftrightarrow \tilde{\chi}$ we get (denoting $x_{ij} \equiv x_i - x_j$)

$$\tilde{L}_\chi^\tau \circ G_\chi = \frac{\gamma_\chi}{N_{\tilde{\chi}}} \tilde{L}_{\tilde{\chi}}^\tau, \quad (1.103)$$

which applied to $f \in C_{\tilde{\chi}}$ gives

$$\begin{aligned} (\tilde{L}_\chi^\tau \circ G_\chi f)(x_1, |a|) &= \int dx_2 K_\chi^\tau(x_{12}, |a|) (G_\chi f)(x_2) \\ &= N_\chi \gamma_\chi \int \frac{dx_2 dx_3}{(x_{23}^2)^\Delta} \left(\frac{|a|}{x_{12}^2 + |a|^2} \right)^{d-\Delta} \tilde{D}^\tau(k_{-\frac{x_{12}}{|a|}}) \Pi_\tau^\mu \hat{a}^\mu(r(x_{23})) f(x_3) \end{aligned}$$

$$\begin{aligned}
 &= N_\chi \gamma_\chi \int \frac{dx_2 dx_3}{(x_{23}^2)^\Delta} \left(\frac{|a|}{x_{12}^2 + |a|^2} \right)^{d-\Delta} \tilde{D}^\tau(k_{-\frac{x_{12}}{|a|}} r(x_{23})) \Pi_\tau^{\tilde{\mu}} f(x_3) \\
 &= \gamma_\chi \int dx_3 \left(\frac{|a|}{x_{13}^2 + |a|^2} \right)^\Delta \tilde{D}^\tau(k_{-\frac{x_{13}}{|a|}}) \Pi_\tau^{\tilde{\mu}} f(x_3) \\
 &= \frac{\gamma_\chi}{N_\chi} \int dx_3 K_\chi^\tau(x_{13}, |a|) f(x_3) = \frac{\gamma_\chi}{N_\chi} (\tilde{L}_\chi^\tau f)(x_1, |a|). \tag{1.104}
 \end{aligned}$$

Finally, we notice that all operators that we have used may be found on the following commutative diagram:

$$\begin{array}{ccc}
 & \hat{C}_\chi^\tau = \hat{C}_\chi^\tau[\phi] & \\
 L_\chi^\tau \swarrow \nearrow \tilde{L}_\chi^\tau & & \tilde{L}_\chi^\tau \nwarrow \searrow L_\chi^\tau \\
 & G_\chi & \\
 C_\chi[\phi_0] & \begin{array}{c} \xleftrightarrow{G_\chi} \\ \xleftarrow{G_\chi} \end{array} & C_\chi[\mathcal{O}, \psi_0]
 \end{array} \tag{1.105}$$

1.2 Intertwining operator realization of anti de Sitter holography

1.2.1 Preliminaries

Lie algebra and group

This section is based on [7]. We consider another example of holography, namely, the Minkowskian AdS/CFT correspondence using the anti de Sitter group $SO(3, 2)$.

We need some well-known preliminaries to set up our notation and conventions. The Lie algebra $\mathcal{G} = \mathfrak{so}(3, 2)$ may be defined as the set of 5×5 matrices X which fulfil the relation:⁷

$${}^t X \eta + \eta X = 0, \tag{1.106}$$

where the metric η is given by

$$\eta = (\eta_{AB}) = \text{diag}(-1, 1, 1, 1, -1), \quad A, B = 0, 1, \dots, 4. \tag{1.107}$$

Then we can choose a basis $X_{AB} = -X_{BA}$ of $\mathcal{G}$ satisfying the commutation relations

$$[X_{AB}, X_{CD}] = \eta_{AC} X_{BD} + \eta_{BD} X_{AC} - \eta_{AD} X_{BC} - \eta_{BC} X_{AD}. \tag{1.108}$$

⁷ For other purposes it may be more convenient to use the other fundamental representation in terms of 4×4 matrices as in [169].

We list the important subalgebras of $\mathcal{G}$:

- $\mathcal{K} = \mathfrak{so}(3) \oplus \mathfrak{so}(2)$, generators: X_{AB} : $(A, B) \in \{1, 2, 3\}, \{0, 4\}$, maximal compact subalgebra;
- $\mathcal{Q}$, generators: X_{AB} : $A \in \{1, 2, 3\}, B \in \{0, 4\}$, noncompact completion of $\mathcal{K}$;
- $\mathcal{A} = \mathfrak{so}(1, 1)$, generator: $D \doteq X_{34}$, dilatations;
- $\mathcal{M} = \mathfrak{so}(2, 1)$, generators: X_{AB} : $(A, B) \in \{0, 1, 2\}$, Lorentz subalgebra;
- $\mathcal{N}$, generators: $T_\mu = X_{\mu 3} + X_{\mu 4}$, $\mu = 0, 1, 2$, translations;
- $\tilde{\mathcal{N}}$, generators: $C_\mu = X_{\mu 3} - X_{\mu 4}$, $\mu = 0, 1, 2$, special conformal transformations;
- $\mathcal{H}$, generators: D, X_{12} , Cartan subalgebra of $\mathcal{G}$.

Thus, we have several decompositions:

- $\mathcal{G} = \mathcal{K} \oplus \mathcal{Q}$, Cartan decomposition;
- $\mathcal{G} = \mathcal{K} \oplus \mathcal{A} \oplus \mathcal{N}$, and $\mathcal{N} \rightarrow \tilde{\mathcal{N}}$, Iwasawa decomposition;
- $\mathcal{G} = \mathcal{N} \oplus \mathcal{M} \oplus \mathcal{A} \oplus \tilde{\mathcal{N}}$, Bruhat decomposition.

The subalgebra $\mathcal{P} = \mathcal{M} \oplus \mathcal{A} \oplus \tilde{\mathcal{N}}$ is a maximal parabolic subalgebra of $\mathcal{G}$.

Finally, we introduce the corresponding connected Lie groups:

$G = \mathrm{SO}_0(3, 2)$ with Lie algebra $\mathcal{G} = \mathfrak{so}(3, 2)$, $K = \mathrm{SO}(3) \times \mathrm{SO}(2)$ is the maximal compact subgroup of G , $A = \exp(\mathcal{A}) = \mathrm{SO}_0(1, 1)$ is abelian simply connected, $N = \exp(\mathcal{N}) \cong \tilde{N} = \exp(\tilde{\mathcal{N}})$, are abelian simply connected subgroups of G preserved by the action of A . The group $M \cong \mathrm{SO}_0(2, 1)$ (with Lie algebra $\mathcal{M}$) commutes with A . The subgroup $P = MAN$ is a *maximal parabolic subgroup* of G . Parabolic subgroups are important because the representations induced from them generate all admissible irreducible representations of semisimple groups [405, 391].

Elementary representations

We use the approach of [161] which we adapt in a condensed form here (see also Volume 1). We work with so-called *elementary representations* (ERs). They are induced from representations of $P = MAN$, where we use finite-dimensional representations of spin $s \in \frac{1}{2}\mathbb{Z}_+$ of M , (non-unitary) characters of A represented by the conformal weight Δ , and the factor N is represented trivially. The data s, Δ is enough to determine a weight $\Lambda \in \mathcal{H}^*$; cf. [161]. Thus, we shall denote the ERs by C^Λ . Sometimes we shall write $\Lambda = [s, \Delta]$. The representation spaces are C^∞ functions on G/P , or equivalently, on the locally isomorphic group $\tilde{N}$ with appropriate asymptotic conditions (which we do not need explicitly; cf. e. g. [184]). We recall that $\tilde{N}$ is isomorphic to 3D Minkowski space-time $\mathfrak{M}$ whose elements will be denoted by $x = (x_0, x_1, x_2)$, while the corresponding elements of $\tilde{N}$ will be denoted by $\tilde{n}_x$. The Lorentzian inner product in $\mathfrak{M}$ is defined as usual:

$$\langle x, x' \rangle \doteq x_0 x'_0 - x_1 x'_1 - x_2 x'_2, \quad (1.109)$$

and we use the notation $x^2 \doteq \langle x, x \rangle$.

The representation action is given as follows:

$$(T^\Lambda(g)\varphi)(x) = y^{-\Delta} D^s(m)\varphi(x') \quad (1.110)$$

the various factors being defined from the local Bruhat decomposition $G \cong_{\text{loc}} \tilde{N}AMN$:

$$g^{-1}\tilde{n}_x = \tilde{n}_{x'}a^{-1}m^{-1}n^{-1}, \quad (1.111)$$

where $y \in \mathbb{R}_+$ parametrizes the elements $a \in A$, $m \in M$, $D^s(m)$ denotes the representation action of M , $n \in N$.

In the above general definition $\varphi(x)$ are considered as elements of the finite-dimensional representation space V^s in which act the operators $D^s(m)$. Following [183, 161] we use scalar functions over an extended space $\mathfrak{M} \times \mathfrak{M}_0$, where $\mathfrak{M}_0$ is a cone parametrized by the variable $\bar{z} = (\bar{z}_0, \bar{z}_1, \bar{z}_2)$ subject to the condition

$$\bar{z}^2 = \langle \bar{z}, \bar{z} \rangle = \bar{z}_0^2 - \bar{z}_1^2 - \bar{z}_2^2 = 0.$$

The internal variable $\bar{z}$ will carry the representation D^s .

The functions on the extended space will be denoted as $\varphi(x, \bar{z})$. On these functions the infinitesimal action of our representations looks as follows:

$$\begin{aligned} T_\mu &= \partial_\mu, \quad \partial_\mu \doteq \frac{\partial}{\partial x_\mu}, \quad \mu = 0, 1, 2 \\ D &= - \sum_{\mu=0}^2 x_\mu \partial_\mu - \Delta, \\ X_{01} &= x_0 \partial_1 + x_1 \partial_0 + \mathfrak{s}_{01}, \quad X_{02} = x_0 \partial_2 + x_2 \partial_0 + \mathfrak{s}_{02}, \\ X_{12} &= -x_1 \partial_2 + x_2 \partial_1 + \mathfrak{s}_{12}, \\ C_0 &= 2x_0 D + x^2 \partial_0 - 2(x_1 \mathfrak{s}_{01} + x_2 \mathfrak{s}_{02}), \\ C_1 &= -2x_1 D + x^2 \partial_1 + 2(x_0 \mathfrak{s}_{01} - x_2 \mathfrak{s}_{12}), \\ C_2 &= -2x_2 D + x^2 \partial_2 + 2(x_0 \mathfrak{s}_{02} + x_1 \mathfrak{s}_{12}), \end{aligned} \quad (1.112)$$

where

$$\mathfrak{s}_{01} = \bar{z}_0 \frac{\partial}{\partial \bar{z}_1} + \bar{z}_1 \frac{\partial}{\partial \bar{z}_0}, \quad \mathfrak{s}_{02} = \bar{z}_0 \frac{\partial}{\partial \bar{z}_2} + \bar{z}_2 \frac{\partial}{\partial \bar{z}_0}, \quad \mathfrak{s}_{12} = -\bar{z}_1 \frac{\partial}{\partial \bar{z}_2} + \bar{z}_2 \frac{\partial}{\partial \bar{z}_1}, \quad (1.113)$$

and they satisfy the $\mathcal{M} = \text{so}(1, 2)$ commutation relations

$$[\mathfrak{s}_{01}, \mathfrak{s}_{02}] = -\mathfrak{s}_{12}, \quad [\mathfrak{s}_{02}, \mathfrak{s}_{12}] = \mathfrak{s}_{01}, \quad [\mathfrak{s}_{12}, \mathfrak{s}_{01}] = \mathfrak{s}_{02}. \quad (1.114)$$

The Casimir operator of $\mathcal{G}$ is given by

$$C = \frac{1}{2} X_{AB} X^{AB} = -X_{01}^2 - X_{02}^2 + X_{12}^2 - D^2 - 3D - C_0 T_0 + C_1 T_1 + C_2 T_2 \quad (1.115)$$

and it is constant on our representation C^Λ :

$$C\varphi = -(\Delta(\Delta - 3) + s(s + 1))\varphi = \lambda(s, \Delta)\varphi. \quad (1.116)$$

Note that the constant $\lambda(s, \Delta)$ has the same value if we replace Δ by $3 - \Delta$. This means that the two boundary (shadow) fields with conformal weights Δ and $3 - \Delta$ are related, or in mathematical language, that the corresponding representations are (partially) equivalent.

Bulk representations

It is well known that the group $SO(3, 2)$ is called also the anti de Sitter group, as it is the isometry group of 4D anti de Sitter space:

$$\xi^A \xi^B \eta_{AB} = 1. \quad (1.117)$$

There are several ways to parametrize anti de Sitter space. We shall utilize the same local Bruhat decomposition as we used in the previous subsection. Thus, we use the local coordinates on the factor space $G/MN \cong_{\text{loc}} \tilde{N}A$, i. e., the coordinates $(x, y) = (x_0, x_1, x_2, y)$, $y \in \mathbb{R}_+$. In this setting anti de Sitter space is called the *bulk space*, while 3D Minkowski space-time is called the *boundary space*, as it is identified with the bulk-boundary value $y = 0$. The functions on the bulk extended with the cone $\mathfrak{M}_0$ will be denoted by $\phi(x, y, \bar{z})$.

As we explained in Section 1.1 we first concentrate on the holography principle, or boundary-to-bulk correspondence, which means to have an operator which maps a boundary field φ to a bulk field ϕ . This map must be invariant w. r. t. the Lie algebra $\mathfrak{so}(3, 2)$. In particular, this means that the Casimir must have the same values in the boundary and bulk representations. The Casimir operator on the boundary representation C^Λ is a constant $\lambda(s, \Delta)$ given in (1.116). Clearly, the (partially) equivalent bulk representation $\hat{C}^\Lambda$ will consist only of functions on which the Casimir operator has the same value.

To give more precisely the $\hat{C}^\Lambda$ we first give a vector-field realization of $\mathfrak{so}(3, 2)$ on the bulk functions $\phi(x, y, \bar{z})$:

$$\begin{aligned} T_\mu &= \partial_\mu, \quad \mu = 0, 1, 2, \\ D &= -\sum_{\mu=0}^2 x_\mu \partial_\mu - y \partial_y, \\ X_{01} &= x_0 \partial_1 + x_1 \partial_0 + \mathfrak{s}_{01}, \quad X_{02} = x_0 \partial_2 + x_2 \partial_0 + \mathfrak{s}_{02}, \\ X_{12} &= -x_1 \partial_2 + x_2 \partial_1 + \mathfrak{s}_{12}, \\ C_0 &= 2x_0 D + (x^2 + y^2) \partial_0 + 2(y \mathfrak{s}_{12} - x_1 \mathfrak{s}_{01} - x_2 \mathfrak{s}_{02}), \\ C_1 &= -2x_1 D + (x^2 + y^2) \partial_1 + 2(y \mathfrak{s}_{02} + x_0 \mathfrak{s}_{01} - x_2 \mathfrak{s}_{12}), \\ C_2 &= -2x_2 D + (x^2 + y^2) \partial_2 + 2(-y \mathfrak{s}_{01} + x_0 \mathfrak{s}_{02} + x_1 \mathfrak{s}_{12}). \end{aligned} \quad (1.118)$$

One may verify by straightforward but lengthy computation that (1.118) satisfies (1.108).

Note that the realization of $\text{so}(3, 2)$ on the boundary given in (1.112) may be obtained from (1.118) by replacing $y\partial_y \rightarrow \Delta$ and then taking the limit $y \rightarrow 0$.

Now we find that the Casimir operator is given in the bulk as follows:

$$C = C_B + C_I - 2y(\mathfrak{s}_{12}\partial_0 - \mathfrak{s}_{02}\partial_1 + \mathfrak{s}_{01}\partial_2), \quad (1.119)$$

$$C_B = y^2(-\partial_0^2 + \partial_1^2 + \partial_2^2) - y^2\partial_y^2 + 2y\partial_y, \quad (1.120)$$

$$C_I = (-\mathfrak{s}_{01}^2 - \mathfrak{s}_{02}^2 + \mathfrak{s}_{12}^2) \quad (1.121)$$

where C_I is the Casimir operator of $\text{so}(1, 2)$ in terms of the internal variables.

Since the Casimir operator in the bulk is not constant but a differential operator, our representation functions will be found as the Casimir eigenfunctions in the bulk. Thus, we consider the eigenvalue problem of the Casimir operator of $\text{so}(3, 2)$:

$$C\phi(x, y, \bar{z}) = \lambda(s, \Delta)\phi(x, y, \bar{z}), \quad \phi \in \widehat{C}^\Lambda. \quad (1.122)$$

In addition, the elements of $\widehat{C}^\Lambda$ must fulfil the appropriate boundary condition:

$$\phi(x, y, \bar{z})|_{y \rightarrow 0} \longrightarrow y^\Lambda \phi(x, 0, \bar{z}), \quad \phi \in \widehat{C}^\Lambda. \quad (1.123)$$

Later we shall see that the elements of $\widehat{C}^\Lambda$ fulfil also the boundary condition with $\Delta \rightarrow 3 - \Delta$ which is natural having in mind the degeneracy of Casimir values for (partially) equivalent representations ($\Delta \rightarrow 3 - \Delta$).

Next we mention that the realization (1.118) causes the infinitesimal transformation of the bulk coordinates:

$$\begin{aligned} T_\mu : x_\mu &\rightarrow x_\mu + a, \\ D : x_\mu &\rightarrow (1 - a)x_\mu, \quad y \rightarrow (1 - a)y, \\ X_{0\mu} : x_0 &\rightarrow x_0 + ax_\mu, \quad x_\mu \rightarrow x_\mu + ax_0, \quad \mu = 1, 2 \\ X_{12} : x_1 &\rightarrow x_1 + ax_2, \quad x_2 \rightarrow x_2 - ax_1, \\ C_0 : x_0 &\rightarrow x_0 + a(y^2 - x_0^2 - x_1^2 - x_2^2), \quad x_{1,2} \rightarrow (1 - 2ax_0)x_{1,2}, \\ &y \rightarrow (1 - 2ax_0)y, \\ C_1 : x_1 &\rightarrow x_1 + a(y^2 + x_0^2 + x_1^2 - x_2^2), \quad x_{0,2} \rightarrow (1 + 2ax_1)x_{0,2}, \\ &y \rightarrow (1 + 2ax_1)y, \\ C_2 : x_2 &\rightarrow x_2 + a(y^2 + x_0^2 - x_1^2 + x_2^2), \quad x_{0,1} \rightarrow (1 + 2ax_2)x_{0,1}, \\ &y \rightarrow (1 + 2ax_2)y. \end{aligned}$$

It follows that every $\text{SO}(3, 2)$ invariant of the two points (x_μ, y) and (x'_μ, y') is a function of

$$u = \frac{4yy'}{(x - x')^2 + (y + y')^2}. \quad (1.124)$$

We set $x'_\mu = 0$, $y' = 1$ and obtain a one-point variable⁸ which shall be very useful below:

$$\hat{u} = \frac{4y}{x^2 + (y+1)^2}. \quad (1.125)$$

1.2.2 Eigenvalue problem and two-point functions in the bulk

Eigenvalue problem of Casimir in the bulk

Here we first solve the equation:

$$C\Psi(x, y, \bar{z}) = \lambda(s, \Delta)\Psi(x, y, \bar{z}). \quad (1.126)$$

We are interested in solutions in which the $\bar{z}$ -dependence is factored out in the form

$$\Psi = \psi(x, y)Q(x, y, \bar{z})^s$$

where Q is homogeneous in $\bar{z}$ of first degree (which is due to the fact that Ψ is homogeneous in $\bar{z}$ of degree $s \in \mathbb{Z}_+$). We assume that ψ is a $SO(3, 2)$ invariant, thus it is function only of $\hat{u}$: $\psi(x, y) = \psi(\hat{u})$. When C_B acts on ψ one may write C_B in terms of $\hat{u}$ only:

$$C_B = \hat{u}^2(\hat{u} - 1)\frac{d^2}{d\hat{u}^2} + 2\hat{u}\frac{d}{d\hat{u}}. \quad (1.127)$$

Furthermore we require that Q is an eigenfunction of C_I . This will guarantee that the spin part Q^s is the eigenfunction of C_I with the correct spin value. It follows that Ψ is also an eigenfunction of C_I :

$$C_I\Psi = \lambda_I\Psi = -s(s+1)\Psi. \quad (1.128)$$

With the fixed vector $(\bar{z}'_0, \bar{z}'_1, \bar{z}'_2)$ in the internal space, we use the following Ansatz for Q

$$Q = \frac{2I_1 - (x^2 - (y+1)^2)I_2 - 2(y+1)I_3}{x^2 + (y+1)^2}, \quad (1.129)$$

$$I_1 = \langle x, \bar{z} \rangle \langle x, \bar{z}' \rangle, \quad I_2 = \langle \bar{z}, \bar{z}' \rangle, \quad I_3 = \sum_{\mu=0}^2 x_\mu (\bar{z} \times \bar{z}')_\mu,$$

where $\bar{z} \times \bar{z}'$ is the standard vector product. (Note that I_1, I_2, I_3 are the three possible scalars which are homogeneous of first degree in both $\bar{z}$ and $\bar{z}'$.) It is easy to verify that

$$C_I I_k = -2I_k. \quad (1.130)$$

⁸ Sometimes in the literature it is called colloquially ‘one-point invariant’.

Thus, we have

$$C_I Q = -2Q \quad (1.131)$$

and one may verify that $C_I Q^s = -s(s+1)Q^s$. With this form of Q the eigenvalue problem is reduced to the second order differential equation:

$$\left((\hat{u} - 1)\hat{u}^2 \frac{d^2}{d\hat{u}^2} + 2\hat{u} \frac{d}{d\hat{u}} - s(s+1)\hat{u} \right) \psi(\hat{u}) = (\lambda - \lambda_I) \psi(\hat{u}) = \Delta(3 - \Delta) \psi(\hat{u}). \quad (1.132)$$

We sketch the derivation of (1.132). First we observe that

$$(C - C_I)\Psi = \Psi[\psi^{-1}C_B\psi + sQ^{-1}(C - C_I)Q + 2s\psi^{-1}Q^{-1}A_1 + s(s-1)Q^{-2}A_2], \quad (1.133)$$

where

$$A_1 = -y^2(\langle \partial\psi, \partial Q \rangle + (\partial_y\psi)(\partial_y Q)) - y((\partial_0\psi)s_{12}Q - (\partial_1\psi)s_{02}Q + (\partial_2\psi)s_{01}Q),$$

$$A_2 = -y^2(\langle \partial Q, \partial Q \rangle + (\partial_y Q)^2) - 2y((\partial_0 Q)s_{12}Q - (\partial_1 Q)s_{02}Q + (\partial_2 Q)s_{01}Q),$$

where $\langle \partial\psi, \partial Q \rangle = (\partial_0\psi)\partial_0 Q - (\partial_1\psi)\partial_1 Q - (\partial_2\psi)\partial_2 Q$. Straightforward computation shows that

$$(C - C_I)Q = -2\hat{u}Q, \quad A_1 = 0, \quad A_2 = -\hat{u}Q^2.$$

Then (1.132) follows from these and (1.127).

Note that although the derivation is different, equation (1.132) is the same as (7.37) of [183] if we make the change $\Delta = \ell + 2$.

Two-point Green function in bulk

We need also the two-point Green function in bulk. Standardly for this we derive the Green function of the operator $C - \lambda$

$$(C - \lambda)\Gamma(x, y, \bar{z}; x', y', \bar{z}') = y^4 \delta^3(x - x') \delta(y - y') (\bar{z}, \bar{z}')^s. \quad (1.134)$$

The computation of Γ is similar to the solution of the eigenvalue problem of C in the previous subsection. We assume Γ has a factored form

$$\Gamma(x, y, \bar{z}; x', y', \bar{z}') = f(u) Q(x, y, \bar{z}; x', y', \bar{z}')^s,$$

where u is the $SO(3, 2)$ invariant of two points (x, y) and (x', y') given in (1.124). This assumption is justified a posteriori since the two-point function is unique up to multiplicative constant.

Note that Γ is an eigenfunction of C_I , i. e., $C_I \Gamma = \lambda_I \Gamma$. Then Γ is given by

$$\begin{aligned}\Gamma &= u^\Delta F(u) Q^s, \\ Q &= \frac{2I'_1 - ((x - x')^2 - (y + y')^2)I_2 - 2(y + y')I'_3}{(x - x')^2 + (y + y')^2}, \\ I'_1 &= \langle x - x', \bar{z} \rangle \langle x - x', \bar{z}' \rangle, \quad I_2 = \langle \bar{z}, \bar{z}' \rangle, \quad I'_3 = \sum_{\mu=0}^2 (x_\mu - x'_\mu)(\bar{z} \times \bar{z}')_\mu,\end{aligned}\tag{1.135}$$

and $F(u)$ is a singular solution of the hypergeometric equation

$$\left(u(1-u) \frac{d^2}{du^2} + 2[\Delta - 1 - \Delta u] \frac{d}{du} + (s - \Delta + 1)(s + \Delta) \right) F(u) = 0. \tag{1.136}$$

We sketch the derivation of the Green function. First we observe that (cf. (1.133))

$$(C - C_I)\Gamma = \Gamma[f^{-1}C_B f + sQ^{-1}(C - C_I)Q + 2sf^{-1}Q^{-1}A_1 + s(s-1)Q^{-2}A_2], \tag{1.137}$$

where

$$\begin{aligned}A_1 &= -y^2(\langle \partial f, \partial Q \rangle + (\partial_y f)(\partial_y Q)) - y((\partial_0 f)\mathfrak{s}_{12}Q - (\partial_1 f)\mathfrak{s}_{02}Q + (\partial_2 f)\mathfrak{s}_{01}Q), \\ A_2 &= -y^2(\langle \partial Q, \partial Q \rangle + (\partial_y Q)^2) - 2y((\partial_0 Q)\mathfrak{s}_{12}Q - (\partial_1 Q)\mathfrak{s}_{02}Q + (\partial_2 Q)\mathfrak{s}_{01}Q).\end{aligned}$$

Straightforward computation shows that

$$(C - C_I)Q = -2uQ, \quad A_1 = 0, \quad A_2 = -uQ^2.$$

It follows that

$$(C - C_I)\Gamma = Q^s \left(u^2(u-1) \frac{d^2}{du^2} + 2u \frac{d}{du} - s(s+1)u \right) f(u).$$

Setting $f(u) = u^\Delta F(u)$ we have

$$\begin{aligned}(C - \lambda)\Gamma &= -Q^s u^{\Delta+1} \left(u(1-u) \frac{d^2}{du^2} + 2[\Delta - 1 - \Delta u] \frac{d}{du} \right. \\ &\quad \left. + (s - \Delta + 1)(s + \Delta) \right) F(u).\end{aligned}\tag{1.138}$$

Thus if $F(u)$ is a singular solution of the hypergeometric equation, then we obtain the RHS of (1.134). The delta functions in the RHS correspond to the singularity at $u = 1 \Leftrightarrow x_\mu = x'_\mu, y = y'$.

Remark 4. One may wonder whether the above may be generalized to the anti de Sitter algebra $\mathfrak{so}(d, 2)$ for $d > 3$. Actually, the only difficulty for $d > 3$ and nontrivial spin would be to find the explicit form of the function Q . In fact, below, some calculations are valid implicitly or explicitly for arbitrary d . $\diamond$

1.2.3 Bulk–boundary correspondence

Consider the fields $\varphi \in C^\Lambda$ and $\phi \in \hat{C}^\Lambda$ from the boundary and bulk representations for the same Λ . By construction they are eigenfunctions of the Casimir operator with the same eigenvalue:

$$\mathcal{C}\varphi = \lambda\varphi, \quad \mathcal{C}\phi = \lambda\phi. \quad (1.139)$$

The bulk field behaves as in (1.123) when approaching the boundary. Thus, we define the bulk-to-boundary operator L_Δ by

$$\begin{aligned} L_\Delta : \hat{C}^\Lambda &\longrightarrow C^\Lambda, \\ \varphi(x, \bar{z}) &= (L_\Delta \phi)(x, \bar{z}) := \lim_{y \rightarrow 0} y^{-\Lambda} \phi(x, y, \bar{z}). \end{aligned} \quad (1.140)$$

On the other hand the boundary-to-bulk operator $\tilde{L}_\Lambda$ is defined by

$$\begin{aligned} \tilde{L}_\Lambda : C^\Lambda &\longrightarrow \hat{C}^\Lambda, \\ \phi(x, y, \bar{z}) &= (\tilde{L}_\Lambda \varphi)(x, \bar{z}) := \int S_\Lambda(x - x', y; \bar{z}, \partial_{\bar{z}'}) \varphi(x', \bar{z}') d^3 x', \end{aligned} \quad (1.141)$$

where the kernel S_Λ is obtained from the two-point Green function Γ defined in (1.134) as follows:

$$S_\Lambda(x - x', y; \bar{z}, \partial_{\bar{z}'}) = \lim_{y' \rightarrow 0} y'^{\Lambda-3} \Gamma(x, y, \bar{z}; x', y', \partial_{\bar{z}'}). \quad (1.142)$$

The formula for S_Λ (for $s \in \mathbb{Z}_+$) is given by

$$S_\Lambda = N_\Lambda \tilde{u}^{3-\Lambda} R^s, \quad \tilde{u} = \frac{4y}{(x - x')^2 + y^2}, \quad R = \frac{\tilde{u}\mathcal{L}}{4y}, \quad (1.143)$$

with

$$\mathcal{L} = 2\tilde{I}_1 - ((x - x')^2 - y^2)\tilde{I}_2 - 2y\tilde{I}_3, \quad (1.144)$$

$$\tilde{I}_1 = \langle x - x', \bar{z} \rangle \langle x - x', \partial_{\bar{z}'} \rangle, \quad \tilde{I}_2 = \langle \bar{z}, \partial_{\bar{z}'} \rangle, \quad \tilde{I}_3 = \sum_{\mu=0}^2 (x_\mu - x'_\mu) (\bar{z} \times \partial_{\bar{z}'})_\mu,$$

and N_Λ is a normalization constant depending on the representation $\Lambda = [s, \Delta]$.

Now we check consistency of the operators L_Δ and $\tilde{L}_\Lambda$:

$$L_\Delta \circ \tilde{L}_\Lambda = \mathbf{1}_{C^\Lambda}, \quad \tilde{L}_\Lambda \circ L_\Delta = \mathbf{1}_{\hat{C}^\Lambda}. \quad (1.145)$$

For the first relation in (1.145) we have to show that

$$\begin{aligned} \varphi(x, \bar{z}) &= (L_\Delta \circ \tilde{L}_\Lambda \varphi)(x, \bar{z}) \\ &= \lim_{y \rightarrow 0} y^{-\Lambda} \int S_\Lambda(x - x', y; \bar{z}, \partial_{\bar{z}'}) \varphi(x', \bar{z}') d^3 x'. \end{aligned} \quad (1.146)$$

We take the limit first by exchanging it and the integral. To calculate the limit it is necessary to express the kernel S_Λ in another form. To this end we establish the following formula of the Fourier transform:

$$\int \frac{e^{i\langle p, X \rangle}}{(\langle X, X \rangle + y^2)^\alpha} \frac{d^3 X}{(2\pi)^{3/2}} = \frac{i\pi}{(-1)^{2\alpha-1} 2^\alpha \Gamma(\alpha)} \left(\frac{\sqrt{-p^2}}{y} \right)^{\alpha-3/2} H_{\alpha-3/2}^{(1)}(y\sqrt{-p^2}), \quad (1.147)$$

where $X_\mu = x_\mu - x'_\mu$ and $H_\beta^{(1)}$ is a Hankel function. The (X_1, X_2) integration can be carried out by making use of the following two formulae: The first one is a formula for $(d-1)$ -dimensional angular integration in d -dimensional Euclidean space:

$$\int f(r) e^{-i\vec{p} \cdot \vec{x}} \frac{d^d x}{(2\pi)^d} = \left(\frac{1}{2\pi} \right)^{d/2} \int_0^\infty r f(r) \left(\frac{r}{p} \right)^{\frac{d}{2}-1} J_{\frac{d}{2}-1}(pr) dr, \quad (1.148)$$

$$\vec{p} = (p_1, p_2, \dots, p_d), \quad \vec{x} = (x_1, x_2, \dots, x_d), \quad p^2 = \sum_{k=1}^d p_k^2, \quad r^2 = \sum_{k=1}^d x_k^2,$$

which is valid for any radial function $f(r)$. The second formula is an integration of the Bessel function:

$$\int_0^\infty \frac{r^{\beta+1} J_\beta(ar)}{(r^2 + \rho^2)^{\gamma+1}} dr = \frac{\alpha^\gamma \rho^{\beta-\gamma} K_{\beta-\gamma}(a\rho)}{2^\gamma \Gamma(\gamma+1)}, \quad 2\Re\gamma + \frac{3}{2} > \Re\beta > -1. \quad (1.149)$$

We modify the second formula (1.149). Set $\beta = 0$ and replace ρ with $-i\rho$, then use the relation between Bessel functions

$$K_\gamma(z) = \frac{\pi}{2} i e^{y\pi i/2} H_\gamma^{(1)}(iz), \quad -\pi < \arg z < \frac{\pi}{2}, \quad (1.150)$$

we obtain

$$\int_0^\infty \frac{r J_0(ar)}{(r^2 - \rho^2)^{\gamma+1}} dr = \frac{i\pi a^\gamma}{(-1)^\gamma 2^{\gamma+1} \Gamma(\gamma+1) \rho^\gamma} H_\gamma^{(1)}(a\rho). \quad (1.151)$$

Now we return to the Fourier transform (1.147). The angular integration in $X_1 X_2$ plane is performed by (1.148) and we use (1.151) for the radial integration in the plane:

$$\begin{aligned} \int \frac{e^{i\langle p, X \rangle}}{(\langle X, X \rangle + y^2)^\alpha} \frac{d^3 X}{(2\pi)^{3/2}} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty dX_0 \int_0^\infty \frac{r J_0(\tilde{p}r) e^{ip_0 X_0} dr}{(-1)^\alpha (r^2 - X_0^2 - y^2)^\alpha} \\ &= \frac{i\pi}{(-1)^{2\alpha-1} \sqrt{2\pi} \Gamma(\alpha)} \left(\frac{\tilde{p}}{2} \right)^{\alpha-1} \int_0^\infty dX_0 \frac{H_{\alpha-1}^{(1)}(\tilde{p} \sqrt{X_0^2 + y^2})}{(X_0^2 + y^2)^{(\alpha-1)/2}} \cos p_0 X_0 \end{aligned}$$

where

$$r^2 = X_1^2 + X_2^2, \quad \tilde{p}^2 = p_1^2 + p_2^2.$$

Recalling that

$$H_\beta^{(1)}(z) = J_\beta(z) + iY_\beta(z)$$

X_0 integration is performed by the formulae of the Fourier cosine transform [210],

$f(r)$	$\int_0^\infty f(r) \cos(rp) dr,$
$\frac{J_\beta(a\sqrt{r^2+b^2})}{(r^2+b^2)^{\beta/2}}$	$\begin{cases} \sqrt{\frac{\pi b}{2}} \frac{(a^2-p^2)^{\beta/2-1/4}}{(ab)^\beta} J_{\beta-1/2}(b\sqrt{a^2-p^2}) & 0 < p < a, \\ 0 & a < p, \end{cases}$
$\frac{Y_\beta(a\sqrt{r^2+b^2})}{(r^2+b^2)^{\beta/2}}$	$\begin{cases} \sqrt{\frac{\pi b}{2}} \frac{(a^2-p^2)^{\beta/2-1/4}}{(ab)^\beta} Y_{\beta-1/2}(b\sqrt{a^2-p^2}) & 0 < p < a, \\ -\sqrt{\frac{2b}{\pi}} \frac{(p^2-a^2)^{\beta/2-1/4}}{(ab)^\beta} K_{\beta-1/2}(b\sqrt{p^2-a^2}) & a < p, \end{cases}$

$\Re\beta > -\frac{1}{2}, \quad a, b > 0.$

By these formulae we obtain

$$\begin{aligned}
 & \int_0^\infty dX_0 \frac{H_{\alpha-1}^{(1)}(\tilde{p}\sqrt{X_0^2+y^2})}{(X_0^2+y^2)^{(\alpha-1)/2}} \cos(p_0 X_0) \\
 &= \begin{cases} \left(\frac{\pi y}{2}\right)^{1/2} \frac{(-\langle p, p \rangle)^{\alpha/2-3/4}}{(\tilde{p}y)^{\alpha-1}} H_{\alpha-3/2}^{(1)}(y\sqrt{\tilde{p}^2-p_0^2}) & 0 < p_0 < \tilde{p} \\ -i\left(\frac{2y}{\pi}\right)^{1/2} \frac{\langle p, p \rangle^{\alpha/2-3/4}}{(\tilde{p}y)^{\alpha-1}} K_{\alpha-3/2}(y\sqrt{p_0^2-\tilde{p}^2}) & \tilde{p} < p_0 \end{cases} \\
 &= \left(\frac{\pi y}{2}\right)^{1/2} \frac{(-\langle p, p \rangle)^{\alpha/2-3/4}}{(\tilde{p}y)^{\alpha-1}} H_{\alpha-3/2}^{(1)}(y\sqrt{\tilde{p}^2-p_0^2}). \tag{1.152}
 \end{aligned}$$

In the last equality equation (1.150) was used to unify two separate cases. Note that $\langle p, p \rangle = p_0^2 - \tilde{p}^2$. In this way the Fourier transform (1.147) has been established.

Now we evaluate the Fourier transform of the kernel S_Λ ,

$$\begin{aligned}
 & \int S_\Lambda(X, y; \bar{z}, \partial_{\bar{z}}) e^{i\langle p, X \rangle} \frac{d^3 X}{(2\pi)^{3/2}} \\
 &= N_\Lambda \int \frac{(4y)^{3-\Delta} S^s e^{i\langle p, X \rangle}}{(\langle X, X \rangle + y^2)^{s-\Delta+3}} \frac{d^3 X}{(2\pi)^{3/2}} \\
 &= \frac{-i\pi N_\Lambda}{2^{s+\Delta-1}\Gamma(s-\Delta+3)y^{s-3/2}} S^s (\sqrt{-p^2})^{s-\Delta+3/2} H_{s-\Delta+3/2}^{(1)}(y\sqrt{-p^2}),
 \end{aligned}$$

where

$$S = -2(\partial_p \cdot \bar{z})\partial_p \cdot \partial_{\bar{z}'} + (\langle \partial_p, \partial_p \rangle + y^2)\langle \bar{z}, \partial_{\bar{z}'} \rangle + 2iy\langle \partial_p, \bar{z} \times \partial_{\bar{z}'} \rangle,$$

with $a \cdot b = \sum_{\mu=0}^2 a^\mu b_\mu$. The inverse Fourier transform gives the following formula of the kernel:

$$S_\Lambda = \frac{-i\pi N_\Lambda}{2^{s+\Delta-1}\Gamma(s-\Delta+3)y^{s-3/2}} \int S^s(\sqrt{-p^2})^{s-\Delta+3/2} \\ \times H_{s-\Delta+3/2}^{(1)}(y\sqrt{-p^2})e^{-i\langle p, X \rangle} \frac{d^3p}{(2\pi)^{3/2}}. \quad (1.153)$$

Since we take a limit of $y \rightarrow 0$, we replace the Hankel function by its asymptotic form

$$-iH_\alpha^{(1)}(z) \rightarrow -\frac{\Gamma(\alpha)}{\pi} \left(\frac{2}{z}\right)^\alpha, \quad z \rightarrow 0.$$

Then

$$-i(\sqrt{-p^2})^{s-\Delta+3/2} H_{s-\Delta+3/2}^{(1)}(y\sqrt{-p^2}) = -\frac{\Gamma(s-\Delta+3/2)}{\pi} \left(\frac{2}{y}\right)^{s-\Delta+3/2}, \\ s-\Delta+3/2 \notin \mathbb{Z}_-,$$

is independent of p_μ so that the action of S is reduced to $y^2 \langle \bar{z}, \partial_{\bar{z}'} \rangle$ and the integration over p becomes Dirac's delta function:

$$S_\Lambda \rightarrow -\frac{(2\pi)^{3/2} N_\Lambda \Gamma(s-\Delta+3/2)}{2^{2\Delta-5/2} \Gamma(s-\Delta+3)} y^\Delta \delta^3(X) \langle \bar{z}, \partial_{\bar{z}'} \rangle^s, \quad (1.154) \\ s-\Delta+3/2 \notin \mathbb{Z}_-, \quad y \rightarrow 0.$$

Substituting this formula of S in (1.146) we obtain

$$\varphi(x, \bar{z}) = -\frac{\pi^{3/2} N_\Lambda \Gamma(s-\Delta+3/2)}{2^{2\Delta-4} \Gamma(s-\Delta+3)} \varphi(x, \bar{z}), \quad (1.155) \\ s-\Delta+3/2 \notin \mathbb{Z}_-, \quad s-\Delta+3 \notin \mathbb{Z}_-.$$

From the latter expression we see the first consistency relation (1.145) to be true by an appropriate choice of N_Λ , e. g.

$$N_\Lambda = -\frac{2^{2\Delta-4} \Gamma(s-\Delta+3)}{\pi^{3/2} \Gamma(s-\Delta+3/2)}, \quad (1.156) \\ s-\Delta+3/2 \notin \mathbb{Z}_-, \quad s-\Delta+3 \notin \mathbb{Z}_-.$$

As a corollary we conclude that for generic values of Δ we can reconstruct a function on anti de Sitter space from its boundary value. Indeed, suppose we have

$$\phi(x, y, \bar{z}) = \int S_\Lambda(x-x', y; \bar{z}, \partial_{\bar{z}'}) f(x', \bar{z}') dx'. \quad (1.157)$$

Then we have for the boundary value

$$\psi_0(x, \bar{z}) \doteq (L_\Delta \phi)(x, \bar{z}) = \lim_{y \rightarrow 0} y^{-\Delta} \phi(x, y, \bar{z}) \quad (1.158)$$

$$= \lim_{y \rightarrow 0} y^{-\Delta} \int S_{\Lambda}(x - x', y; \bar{z}, \partial_{\bar{z}'}) f(x', \bar{z}') dx' = f(x).$$

Now we can prove the second consistency relation in (1.145):

$$\begin{aligned} (\tilde{L}_{\Lambda} \circ L_{\Delta} \phi)(x, y, \bar{z}) &= \int S_{\Lambda}(x - x', y; \bar{z}, \partial_{\bar{z}'}) (L_{\Delta} \phi)(x', \bar{z}') dx' \\ &= \int S_{\Lambda}(x - x', y; \bar{z}, \partial_{\bar{z}'}) \lim_{y' \rightarrow 0} y'^{-\Delta} \phi(x', y', \bar{z}') \\ &= \int S_{\Lambda}(x - x', y; \bar{z}, \partial_{\bar{z}'}) \psi_0(x', \bar{z}') dx' = \phi(x, y, \bar{z}) \end{aligned} \quad (1.159)$$

where in the last line we used (1.158).

1.2.4 Intertwining properties

Here we investigate the intertwining properties of the boundary $\leftrightarrow$ bulk operators.

Bulk-to-boundary operator L_{Δ}

It is not difficult to verify the intertwining property of the bulk-to-boundary operator L_{Δ} . Namely, one should verify the following:

$$L_{\Delta} \circ \hat{X} = \tilde{X} \circ L_{\Delta}, \quad (1.160)$$

where $X \in \mathfrak{so}(3, 2)$, $\tilde{X}$ denotes the action of the generator X on the boundary (1.112) and $\hat{X}$ denotes the action of the generator in the bulk (1.118). More explicitly,

$$\tilde{X} \phi(x, \bar{z}) = \lim_{y \rightarrow 0} y^{-\Delta} \hat{X} \phi(x, y, \bar{z}), \quad \phi \in C^{\Lambda}, \quad \phi \in \tilde{C}^{\Lambda}. \quad (1.161)$$

If the field ϕ belongs to the conjugate representation $\phi \in C^{\tilde{\Lambda}}$, $\tilde{L} = [s, 3 - \Delta]$, then relations (1.160) and (1.161) hold with the change $\Delta \rightarrow 3 - \Delta$, the same change being made also in (1.112).

Boundary-to-bulk operator $\tilde{L}_{\Lambda}$

The intertwining property of the boundary-to-bulk operator $\tilde{L}_{\Lambda}$ means that

$$\hat{X} \circ \tilde{L}_{\Lambda} = \tilde{L}_{\Lambda} \circ \tilde{X}. \quad (1.162)$$

More explicitly, it reads

$$\hat{X} \phi(x, y, \bar{z}) = \int S_{\Lambda}(x, y, \bar{z}; x', \partial_{\bar{z}'}) \tilde{X}_{\Lambda} \phi(x', \bar{z}') d^3 x', \quad \phi \in \tilde{C}^{\Lambda}, \quad \phi \in C^{\Lambda}. \quad (1.163)$$

This is an immediate consequence of $L_\Delta \circ \tilde{L}_\Lambda = \mathbf{1}_{C^\Lambda}$, $\tilde{L}_\Lambda \circ L_\Delta = \mathbf{1}_{\hat{C}^\Lambda}$ and (1.160). By sandwiching (1.160) by $\tilde{L}_\Lambda$ one has

$$\tilde{L}_\Lambda \circ L_\Delta \circ \hat{X} \circ \tilde{L}_\Lambda = \tilde{L}_\Lambda \circ \tilde{X} \circ L_\Delta \circ \tilde{L}_\Lambda, \quad \text{acting on } C^\Lambda$$

This is nothing but (1.162).

For a proof of (1.163) by direct computation, we refer to [7].

Further intertwining relations

We start by recording the second limit of the bulk functions

$$\begin{aligned} \varphi_0(x, \bar{z}) &\doteq \lim_{y \rightarrow 0} y^{\Delta-3} \phi(x, y, \bar{z}) \\ &= \lim_{y \rightarrow 0} y^{\Delta-3} \int S_\Lambda(x - x', y; \bar{z}, \partial_{\bar{z}'}) \psi_0(x', \bar{z}') dx' \\ &= N_\Lambda \lim_{y \rightarrow 0} y^{\Delta-3} \int \left(\frac{4y}{(x - x')^2 + y^2} \right)^{3-\Delta} \\ &\quad \times \left(\frac{\mathcal{L}}{(x - x')^2 + y^2} \right)^s \psi_0(x', \bar{z}') dx' \\ &= N_\Lambda \int \left(\frac{4}{(x - x')^2} \right)^{3-\Delta} \\ &\quad \times \left(\frac{2\tilde{I}_1 - ((x - x')^2)\tilde{I}_2}{(x - x')^2} \right)^s \psi_0(x', \bar{z}') dx' \\ &= \mathcal{N}_\Lambda \int \frac{dx'}{((x - x')^2)^{3-\Delta}} \\ &\quad \times \left(\frac{2\langle x - x', \bar{z} \rangle \langle x - x', \partial_{\bar{z}'} \rangle}{(x - x')^2} - \langle \bar{z}, \partial_{\bar{z}'} \rangle \right)^s \psi_0(x', \bar{z}') \\ &= \mathcal{N}_\Lambda \int \frac{dx'}{((x - x')^2)^{3-\Delta}} (r(x - x'; \bar{z}, \partial_{\bar{z}'}))^s \psi_0(x', \bar{z}') \\ &= \frac{\mathcal{N}_\Lambda}{\gamma_{\tilde{L}}} \int dx' G_{\tilde{L}}(x - x'; \bar{z}, \partial_{\bar{z}'}) \psi_0(x', \bar{z}'), \quad \mathcal{N}_\Lambda = 4^{3-\Delta} N_\Lambda, \end{aligned} \tag{1.164}$$

where in the second line we have used (1.158), in the third line we have used (1.143) and (1.144), and in the last line we have recovered the well-known conformal two-point function, cf., e. g. [495],

$$\begin{aligned} G_\Lambda(x; \bar{z}, \bar{z}') &= \gamma_\Lambda \frac{(r(x; \bar{z}, \bar{z}'))^s}{(x^2)^\Delta}, \\ r(x; \bar{z}, \bar{z}') &= r(x)_{\mu\sigma} \bar{z}^\mu \bar{z}'^\sigma, \quad r(x)_{\mu\sigma} = \frac{2}{x^2} x_\mu x_\sigma - g_{\mu\sigma}, \\ g &= (g_{\mu\nu}) = \text{diag}(1, -1, -1), \end{aligned} \tag{1.165}$$

for the conjugate weight $\tilde{L} = [s, 3 - \Delta]$. The latter is natural since $\psi_0 \in C^\Lambda$, $\varphi_0 \in C^{\tilde{L}}$, and the conformal two-point function realizes the equivalence of the conjugate representations $\Lambda, \tilde{L}$ which have the same Casimir values; cf. [183]. The normalization constant γ_Λ depends on the representation $\Lambda = [s, \Delta]$ and below we derive a formula for the product $\gamma_\Lambda \gamma_{\tilde{L}}$.

Further, using (1.164) we define the operator G_Λ through the kernel $G_\Lambda(x; \bar{z}, \bar{z}')$:

$$\begin{aligned} G_\Lambda : C^{\tilde{L}} &\rightarrow C^\Lambda, \\ (G_\Lambda \varphi_0)(x, \bar{z}) &= \int dx' G_\Lambda(x - x'; \bar{z}, \partial_{\bar{z}'}) \varphi_0(x', \bar{z}'). \end{aligned} \quad (1.166)$$

Then equation (1.164) may be written as

$$L_{\tilde{\Delta}} = \frac{\mathcal{N}_\Lambda}{\gamma_{\tilde{L}}} G_{\tilde{L}} \circ L_\Delta, \quad \tilde{\Delta} \doteq 3 - \Delta \quad (1.167)$$

as operators acting on the bulk representation $\widehat{C}^\Lambda$.

Note that at generic points (those not excluded in (1.155)) the operators G_Λ and $G_{\tilde{L}}$ are inverse to each other [183]:

$$G_\Lambda \circ G_{\tilde{L}} = \mathbf{1}_{C^\Lambda}, \quad G_{\tilde{L}} \circ G_\Lambda = \mathbf{1}_{C^{\tilde{L}}}. \quad (1.168)$$

At generic points from this we can obtain a lot of interesting relations; e. g. applying $\tilde{L}_\Lambda$ from the right we get

$$L_{\tilde{\Delta}} \circ \tilde{L}_\Lambda = \frac{\mathcal{N}_\Lambda}{\gamma_{\tilde{L}}} G_{\tilde{L}}. \quad (1.169)$$

Then we write down the conjugate relation:

$$L_\Delta \circ \tilde{L}_{\tilde{L}} = \frac{\mathcal{N}_{\tilde{L}}}{\gamma_\Lambda} G_\Lambda. \quad (1.170)$$

Then we combine relations (1.169) and (1.170):

$$L_\Delta \circ \tilde{L}_{\tilde{L}} \circ L_{\tilde{\Delta}} \circ \tilde{L}_\Lambda = \frac{\mathcal{N}_{\tilde{L}}}{\gamma_\Lambda} \frac{\mathcal{N}_\Lambda}{\gamma_{\tilde{L}}} G_\Lambda \circ G_{\tilde{L}} = \frac{\mathcal{N}_{\tilde{L}}}{\gamma_\Lambda} \frac{\mathcal{N}_\Lambda}{\gamma_{\tilde{L}}} \mathbf{1}_\Lambda. \quad (1.171)$$

For the LHS of (1.171) we use the second relation of (1.145), then we use the first to obtain

$$L_\Delta \circ \tilde{L}_{\tilde{L}} \circ L_{\tilde{\Delta}} \circ \tilde{L}_\Lambda = L_\Delta \circ \mathbf{1}_{\widehat{C}^\Lambda} \circ \tilde{L}_\Lambda = L_\Delta \circ \tilde{L}_\Lambda = \mathbf{1}_\Lambda. \quad (1.172)$$

Thus, from (1.171) and (1.172) it follows that

$$\begin{aligned} \gamma_\Lambda \gamma_{\tilde{L}} = \mathcal{N}_\Lambda \mathcal{N}_{\tilde{L}} &= \frac{2^4 \Gamma(s - \Delta + 3) \Gamma(s + \Delta)}{\pi^3 \Gamma(s - \Delta + 3/2) \Gamma(s + \Delta - 3/2)}, \\ s - \Delta + 3/2 &\notin \mathbb{Z}_-, \quad s - \Delta + 3 \notin \mathbb{Z}_-, \\ s + \Delta - 3/2 &\notin \mathbb{Z}_-, \quad s + \Delta \notin \mathbb{Z}_-. \end{aligned} \quad (1.173)$$

The product of constants in (1.173) should be proportional to the analytic continuation of the Plancherel measure for the Plancherel formula contribution of the principal series of unitary irreps of G , cf., e. g. [168], but we shall not go into that.

Remark 5. One may wonder what happens at the excluded values in (1.173). This requires further nontrivial examination. Such a study was done in the Euclidean case in [183]; see also Section 1.1. Since some results follow by a Wick rotation, we may conjecture that, for example, the operators G_Λ and $G_{\bar{L}}$ would not be inverse to each other. This would be the case since at these points the representations C^Λ and $C^{\bar{L}}$ would be reducible and the G -operators would have kernels. $\diamond$

1.3 Nonrelativistic holography

This section is based on [6]. We consider our third example of holography, namely, a nonrelativistic case, using the Schrödinger group.

1.3.1 Preliminaries

The role of nonrelativistic symmetries in theoretical physics has always been important. Currently one of the most popular fields in theoretical physics—string theory, which claims to be a universal theory—encompasses relativistic quantum field theory, classical gravity, and certainly, nonrelativistic quantum mechanics, in such a way that it is not even necessary to separate these components.

Since the cornerstone of quantum mechanics is the Schrödinger equation then it is not a surprise that the Schrödinger group—the group that is the maximal group of symmetry of the Schrödinger equation—is playing recently more and more a prominent role in theoretical physics [476, 521, 543, 35, 278, 314, 435, 203, 578, 303, 11, 31, 249, 564, 120, 75, 484, 6, 466, 517, 330, 126, 321, 85, 16, 294, 93, 400, 62, 331, 560, 590, 366]. The latter is natural since originally the Schrödinger group, actually the Schrödinger algebra, was introduced by Niederer [474] and Hagen [298] as a nonrelativistic limit of the vector-field realization of the conformal algebra. We mention some recent developments to nonrelativistic conformal holography; cf., e. g. [218, 219, 4, 465, 557, 467, 10, 577, 21, 200, 587, 280, 201, 94, 575, 36, 126, 309, 547]

Recently, Son [543] proposed another method of identifying the Schrödinger algebra in $d + 1$ space-time. Namely, Son started from anti de Sitter (AdS) space in $d + 3$ -dimensional space-time with metric that is invariant under the corresponding conformal algebra $so(d + 1, 2)$ and then deformed the AdS metric to reduce the symmetry to the Schrödinger algebra.

In view of the relation of the conformal and Schrödinger algebra there arises a natural question: Is there a nonrelativistic analogue of the AdS/CFT correspondence,

in which the conformal symmetry is replaced by the Schrödinger symmetry? Indeed, this is to be expected since the Schrödinger equation should play a role both in the bulk and on the boundary. This section provides a positive answer to this question.

Schrödinger algebra $\hat{S}(n)$

The Schrödinger algebra $S(n)$ ($n \geq 1$), in $(n+1)$ -dimensional space-time has $(n^2 + 3n + 6)/2$ generators with the following nontrivial commutation relations, cf., e. g. Barut-Rączka [43]:

$$[J_{ij}, J_{kl}] = \delta_{ik}J_{jl} + \delta_{jl}J_{ik} - \delta_{il}J_{jk} - \delta_{jk}J_{il}, \quad (1.174a)$$

$$[J_{ij}, P_k] = \delta_{ik}P_j - \delta_{jk}P_i, \quad (1.174b)$$

$$[J_{ij}, G_k] = \delta_{ik}G_j - \delta_{jk}G_i, \quad (1.174c)$$

$$[P_t, G_i] = P_i, \quad (1.174d)$$

$$[K, P_i] = -G_i, \quad (1.174e)$$

$$[D, G_i] = G_i, \quad (1.174f)$$

$$[D, P_i] = -P_i, \quad (1.174g)$$

$$[D, P_t] = -2P_t, \quad (1.174h)$$

$$[D, K] = 2K, \quad (1.174i)$$

$$[P_t, K] = D, \quad (1.174j)$$

where the $J_{ij} = -J_{ji}$, $i, j = 1, 2, \dots, n$, are the generators of the rotation subalgebra $\mathfrak{so}(n)$, P_i , $i = 1, 2, \dots, n$, are the generators of the abelian subalgebra $\mathfrak{t}(n)$ of space translations, G_i , $i = 1, 2, \dots, n$, are the generators of the abelian subalgebra $\mathcal{G}(n)$ of special Galilei transformations, P_t is the generator of time translations, D is the generator of dilatations (scale transformations), and K is the generator of Galilean conformal transformations.

Actually, mostly we shall work with the central extension of the Schrödinger algebra $\hat{S}(n)$, obtained by adding the central element M to $S(n)$ which enters the additional commutation relations:

$$[P_k, G_\ell] = \delta_{k\ell}M. \quad (1.175)$$

Note that the centre is one-dimensional. Of course, (1.175) gives also a central extension of the Galilei subalgebra $\mathcal{G}(n)$; however, for $n = 1, 2$ this is not the full central extension $\hat{\mathcal{G}}(n)$ of $\mathcal{G}(n)$, since the centre is $(n+1)$ -dimensional in these cases; cf., e. g. [43].

The centrally extended Schrödinger algebra for $n = 3$ was introduced in [474, 298] by deformation and extension of the standard vector-field realization of the conformal

algebra $\mathcal{C}(3)$ in $3 + 1$ -dimensional space-time. The resulting vector-field realization for arbitrary n is

$$P_j = \partial_j, \quad (1.176a)$$

$$G_j = t\partial_j + Mx_j, \quad (1.176b)$$

$$P_t = \partial_t, \quad (1.176c)$$

$$D = 2t\partial_t + x_j\partial_j + \Delta, \quad (1.176d)$$

$$K = t^2\partial_t + tx_j\partial_j + \frac{M}{2}x_j^2 + t\Delta, \quad (1.176e)$$

$$J_{jk} = x_k\partial_j - x_j\partial_k, \quad (1.176f)$$

where $\partial_t \equiv \partial/\partial_t$, $\partial_j \equiv \partial/\partial_{x_j}$, summation over repeated indices is assumed, Δ is a number called the conformal weight (more about it will be said below). We note that (1.176f) may be extended by the matrices of a finite-dimensional representation Σ_{jk} of $\mathfrak{so}(n)$ which satisfies (1.174a) as follows:

$$J_{jk} = x_k\partial_j - x_j\partial_k + \Sigma_{jk}. \quad (1.176f')$$

Now we list the important subalgebras of the Schrödinger algebra $\mathcal{S}(n)$:

The generators J_{ij}, P_i form the $((n+1)n/2)$ -dimensional Euclidean subalgebra $\mathcal{E}(n)$.

The generators J_{ij}, P_i, D form the $((n+1)n/2 + 1)$ -dimensional Euclidean Weyl subalgebra $\mathcal{W}(n)$.

The subalgebras $\tilde{\mathcal{E}}(n)$ and $\tilde{\mathcal{W}}(d)$ generated by J_{ij}, G_i and by J_{ij}, G_i, D , respectively, are isomorphic to $\mathcal{E}(n), \mathcal{W}(n)$, respectively.

The generators J_{ij}, P_i, G_i, P_t form the $((n+1)(n+2)/2)$ -dimensional Galilei subalgebra $\mathcal{G}(n)$. The generators J_{ij}, P_i, G_i, K form another $((n+1)(n+2)/2)$ -dimensional subalgebra $\tilde{\mathcal{G}}(n)$ which is isomorphic to the Galilei subalgebra.

The isomorphic pairs mentioned above are conjugate to each other in a sense explained below.

For the structure of $\hat{\mathfrak{s}}(n)$ it is also important to note that the generators D, K, P_t form an $\mathfrak{sl}(2, \mathbb{R})$ subalgebra.

Obviously $\mathcal{S}(n)$ is not semisimple and has the following Levi–Malcev decomposition (for $n \neq 2$):

$$\mathcal{S}(n) = \mathcal{N}(n) \oplus \mathcal{M}(n), \quad (1.177)$$

$$\mathcal{N}(n) = \mathfrak{t}(n) \oplus \mathfrak{g}(n),$$

$$\mathfrak{t}(n) = \text{l.s.}\{P_i\}, \quad \mathfrak{g}(n) = \text{l.s.}\{G_i\},$$

$$\mathcal{M}(n) = \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{so}(n),$$

with $\mathcal{M}(n)$ acting on $\mathcal{N}(n)$, where the maximal solvable ideal $\mathcal{N}(n)$ is abelian, while the semisimple subalgebra (the Levi factor) is $\mathcal{M}(n)$.

For $n = 2$ the maximal solvable ideal $t(n) \oplus g(n) \oplus \mathfrak{so}(2)$ is not abelian, while the Levi factor $\mathfrak{sl}(2, \mathbb{R})$ is simple. Note, however, that many statements below will hold for arbitrary n , including $n = 2$, if we extend the definition of $\mathcal{N}(n)$ and $\mathcal{M}(n)$ to the case $n = 2$ (which is natural since $\mathcal{M}(2)$ is then reductive, and it is well known that many semisimple structural and representation-theoretic results hold for the reductive case).

The commutation relations (1.174) are graded if we define

$$\deg D = 0, \quad (1.178a)$$

$$\deg G_j = 1, \quad (1.178b)$$

$$\deg K = 2, \quad (1.178c)$$

$$\deg P_j = -1, \quad (1.178d)$$

$$\deg P_t = -2, \quad (1.178e)$$

$$\deg M = 0, \quad (1.178f)$$

$$\deg J_{jk} = 0. \quad (1.178g)$$

As expected the corresponding grading operator is D .

For future reference we record also the following involutive antiautomorphism of the Schrödinger algebra:

$$\omega(P_t) = K, \quad \omega(P_j) = G_j, \quad \omega(J_{jk}) = -J_{jk} \quad \omega(D) = D, \quad \omega(M) = M. \quad (1.179)$$

Note that this conjugation is transforming the isomorphic pairs of subalgebras introduced above, namely, we have

$$\omega(\mathcal{E}) = \tilde{\mathcal{E}}, \quad \omega(\mathcal{W}) = \widetilde{\mathcal{W}}, \quad \omega(\mathcal{G}) = \tilde{\mathcal{G}}. \quad (1.180)$$

We end this discussion of the general structure of $\hat{S}(n)$ with the question of invariant scalar products. Since the Schrödinger algebra is not semisimple its Cartan–Killing form is degenerate. More than that—the Schrödinger algebra does not have any nondegenerate ad-invariant symmetric bilinear form [175]. For the discussion of non-semisimple Lie algebras with nondegenerate ad-invariant symmetric bilinear form we refer to [445].

Matrix representation

It is useful to have a representation of $\hat{S}(n)$ by $(2n + 2) \times (2n + 2)$ matrices:

$$\begin{aligned} D_{ab} &= \sum_{\mu=1}^n (-\delta_{a,2\mu} \delta_{b,2\mu} + \delta_{a,2\mu+1} \delta_{b,2\mu+1}), \quad M_{ab} = -2\delta_{a1} \delta_{b,2n+2}, \\ (G_k)_{ab} &= \delta_{a1} \delta_{b,2k} - \delta_{a,2n+3-2k} \delta_{b,2n+2}, \\ (P_k)_{ab} &= \delta_{a1} \delta_{b,2n+3-2k} + \delta_{a,2k} \delta_{b,2n+2}, \end{aligned} \quad (1.181)$$

$$\begin{aligned}
(P_t)_{ab} &= - \sum_{\mu=1}^n \delta_{a,2\mu} \delta_{b,2n+3-2\mu}, \quad (K)_{ab} = \sum_{\mu=1}^n \delta_{a,2\mu+1} \delta_{b,2n+2-2\mu}, \\
(J_{kl})_{ab} &= \delta_{a,2l} \delta_{b,2k} - \delta_{a,2k} \delta_{b,2l} - \delta_{a,2n+3-2k} \delta_{b,2n+3-2l} + \delta_{a,2n+3-2l} \delta_{b,2n+3-2k},
\end{aligned}$$

where X_{ab} denotes the (a, b) element of matrix X . In this representation D is diagonal: $D = \text{diag}(0, -1, 1, -1, 1, \dots, -1, 1, 0)$, while P_t and K are minor-diagonal.

For $n = 3$ we give the above explicitly. The positions of nonzero entries are indicated by the name of the generators as follows:

$$n = 3, \quad \begin{pmatrix} 0 & G_1 & P_3 & G_2 & P_2 & G_3 & P_1 & M \\ & D & & J_{12} & & J_{13} & P_t & P_1 \\ & & D & & J_{23} & K & J_{13} & G_3 \\ & J_{12} & & D & P_t & J_{23} & & P_2 \\ & & J_{23} & K & D & & J_{12} & G_2 \\ & J_{13} & P_t & J_{23} & & D & & P_3 \\ & K & J_{13} & & J_{12} & & D & G_1 \\ 0 & & & & & & & 0 \end{pmatrix}. \quad (1.182)$$

The first column and eighth row are empty.

1.3.2 Triangular decomposition of $\hat{\mathcal{S}}(n)$

The grading (1.178) can be viewed as extension of the triangular decomposition of the algebra $\mathfrak{sl}(2, \mathbb{R}) = \mathfrak{sl}(2, \mathbb{R})^+ \oplus \mathfrak{sl}(2, \mathbb{R})^0 \oplus \mathfrak{sl}(2, \mathbb{R})^-$, where $\mathfrak{sl}(2, \mathbb{R})^+$ is spanned by K , the Cartan subalgebra $\mathfrak{sl}(2, \mathbb{R})^0$ is spanned by D , and $\mathfrak{sl}(2, \mathbb{R})^-$ is spanned by P_t . Taking into account also the triangular decomposition: $\mathfrak{so}(n) = \mathfrak{so}(n)^+ \oplus \mathfrak{so}(n)^0 \oplus \mathfrak{so}(n)^-$ (more precisely of its complexification $\mathfrak{so}(n, \mathbb{C})$), we can introduce the following triangular decomposition:

$$\begin{aligned}
\hat{\mathfrak{s}}(n) &= \hat{\mathfrak{s}}(n)^+ \oplus \hat{\mathfrak{s}}(n)^0 \oplus \hat{\mathfrak{s}}(n)^- \\
\hat{\mathfrak{s}}(n)^+ &= \mathfrak{g}(n) \oplus \mathfrak{sl}(2, \mathbb{R})^+ \oplus \mathfrak{so}(n)^+ \\
\hat{\mathfrak{s}}(n)^0 &= \mathfrak{sl}(2, \mathbb{R})^0 \oplus \mathfrak{so}(n)^0 \oplus \text{lin.span } M, \\
\hat{\mathfrak{s}}(n)^- &= \mathfrak{t}(n) \oplus \mathfrak{sl}(2, \mathbb{R})^- \oplus \mathfrak{so}(n)^-.
\end{aligned} \quad (1.183)$$

(Clearly, for $n = 1$ the $\mathfrak{so}(n)$ factors are missing, while for $n = 2$ only the Cartan subalgebra $\mathfrak{so}(n)^0$ survives.)

1.3.3 Choice of bulk and boundary

In the beginning of this section we review the work of Son [543]. To realize the Schrödinger symmetry in $(n + 1)$ dimensions geometrically, Son takes the AdS metric,

which is invariant under the conformal group $O(n+2, 2)$ in $(n+2)$ dimensions, and then deforms it to reduce the symmetry down to the Schrödinger group. The AdS space, in Poincaré coordinates, is

$$ds^2 = \frac{\eta_{\mu\nu} dx^\mu dx^\nu + dz^2}{z^2}, \quad \mu, \nu = 0, 1, \dots, n+1, \quad \eta_{\mu\nu} = \text{diag}(-1, 1, \dots, 1). \quad (1.184)$$

The generators of the conformal group correspond to the following infinitesimal coordinates transformations that leave the metric unchanged:

$$\begin{aligned} P^\mu : x^\mu &\rightarrow x^\mu + a^\mu, \\ D : x^\mu &\rightarrow (1-a)x^\mu, \quad z \rightarrow (1-a)z, \\ K^\mu : x^\mu &\rightarrow x^\mu + a^\mu(z^2 + x \cdot x) - 2x^\mu(a \cdot x), \end{aligned} \quad (1.185)$$

where $x \cdot x \equiv \eta_{\mu\nu} x^\mu x^\nu$.

Then Son deforms the above metric so as to reduce the symmetry to the Schrödinger group. The resulting metric is [543]

$$ds^2 = -\frac{2(dx^+)^2}{z^4} + \frac{-2dx^+ dx^- + dx^i dx^i + dz^2}{z^2}, \quad i = 1, \dots, n. \quad (1.186)$$

It is straightforward to verify that the metric (1.186) exhibits a full Schrödinger symmetry. Indeed, the generators of the Schrödinger algebra correspond to the following isometries of the metric:

$$\begin{aligned} P_i : x_i &\rightarrow x_i + a_i, \quad H : x_+ \rightarrow x_+ + a, \quad M : x_- \rightarrow x_- + a, \\ G_i : x_i &\rightarrow x_i - a_i x_+, \quad x_- \rightarrow x_- - a_i x_i, \\ D : x_i &\rightarrow (1-a)x_i, \quad z \rightarrow (1-a)z, \quad x_+ \rightarrow (1-a)_2 x_+, \quad x_- \rightarrow x_-, \\ K : z &\rightarrow (1-ax_+)z, \quad x_i \rightarrow (1-ax_+)x_i, \quad x_+ \rightarrow (1-ax_+)x_+, \\ x_- &\rightarrow x_- - \frac{a}{2}(x_i x_i + z^2), \end{aligned} \quad (1.187)$$

while the generators J_{jk} of $\text{so}(n)$ rotate the coordinates x_j as before. We require that the Schrödinger algebra is an isometry of the above metric. We also need to replace the central element M by the derivative of the variable x_- which is chosen so that $\frac{\partial}{\partial x_-}$ continues to be central. Note the variable x_- does not scale w. r. t. D . Such variables are called ultralocal.

Thus, a vector-field realization of the Schrödinger algebra $\hat{\mathcal{S}}(n)$ in the $(n+3)$ -dimensional bulk space (t, x_i, x_-, z) is

$$\begin{aligned} P_j &= \partial_j, \\ G_j &= t\partial_j + mx_j, \\ P_t &= \partial_t, \end{aligned} \quad (1.188)$$

$$\begin{aligned}
D &= 2t\partial_t + x_j\partial_j + z\frac{\partial}{\partial z}, \\
K &= t^2\partial_t + tx_j\partial_j + tz\frac{\partial}{\partial z} + \frac{1}{2}(x_j^2 + z^2)M, \\
J_{jk} &= x_k\partial_j - x_j\partial_k, \\
M &= \frac{\partial}{\partial x_-}.
\end{aligned}$$

We would like to treat the realization (1.176) as a vector-field realization on the boundary of the bulk space (t, x_i, x_-, z) . Obviously, the variable z is the variable distinguishing the bulk, namely, the boundary is obtained when $z = 0$. The exact map will be displayed below but heuristically, passing from (1.188) to (1.176) one first replaces $z\frac{\partial}{\partial z}$ with Δ and then sets $z = 0$.

1.3.4 One-dimensional case

Here and below we review Reference [6]. Now we restrict ourselves to the $1+1$ -dimensional case, $n = 1$. In this case the centrally extended Schrödinger algebra has six generators:

- time translation: P_t
- space translation: P_x
- Galilei boost: G
- dilatation: D
- conformal transformation: K
- mass: M

with the following non-vanishing commutation relations:

$$\begin{aligned}
[P_t, D] &= 2P_t, & [D, K] &= 2K, & [P_t, K] &= D, \\
[P_t, G] &= P_x, & [P_x, D] &= P_x, \\
[P_x, K] &= G, & [D, G] &= G, \\
[P_x, G] &= M.
\end{aligned} \tag{1.189}$$

Further we need also the Casimir operator. It turns out that the lowest order non-trivial Casimir operator is the fourth order one [493]:

$$\tilde{C}_4 = (2MD - \{P_x, G\})^2 - 2\{2MK - G^2, 2MP_t - P_x^2\} \tag{1.190}$$

In fact, there are many cancelations, and the central generator M is a common linear multiple. (This is seen immediately by setting $M = 0$, then $\tilde{C}_4 \rightarrow 0$.)

The metric (1.186) of the four-dimensional bulk space (t, x, x_-, z) now reads⁹

$$ds^2 = -\frac{2(dt)^2}{z^4} + \frac{-2dtdx_- + (dx)^2 + dz^2}{z^2}. \quad (1.191)$$

Accordingly, the vector-field realization of the Schrödinger algebra is given by

$$\begin{aligned} P_t &= \frac{\partial}{\partial t}, & P_x &= \frac{\partial}{\partial x}, & M &= \frac{\partial}{\partial x_-}, \\ G &= t \frac{\partial}{\partial x} + xM, \\ D &= x \frac{\partial}{\partial x} + z \frac{\partial}{\partial z} + 2t \frac{\partial}{\partial t}, \\ K &= t \left(x \frac{\partial}{\partial x} + z \frac{\partial}{\partial z} + t \frac{\partial}{\partial t} \right) + \frac{1}{2}(x^2 + z^2)M, \end{aligned} \quad (1.192)$$

and it generates an isometry of (1.191). This vector-field realization of the Schrödinger algebra acts on the bulk fields $\phi(t, x, x_-, z)$.

In this realization the Casimir operator becomes

$$\begin{aligned} \tilde{C}_4 &= M^2 C_4, \\ C_4 &= \hat{Z}^2 - 4\hat{Z} - 4z^2 \hat{S} = 4z^2 \frac{\partial^2}{\partial z^2} - 8z \partial_z + 5 - 4z^2 \hat{S}, \end{aligned} \quad (1.193)$$

$$\begin{aligned} \hat{S} &\equiv 2\partial_t \partial_{x_-} - \frac{\partial^2}{\partial x^2}, \\ \hat{Z} &\equiv 2z \partial_z - 1. \end{aligned} \quad (1.194)$$

Note that (1.194) is the pro-Schrödinger operator.

The vector-field realization (1.176) of the Schrödinger algebra on the boundary becomes

$$\begin{aligned} P_t &= \frac{\partial}{\partial t}, & P_x &= \frac{\partial}{\partial x}, & M &= \frac{\partial}{\partial x_-}, \\ G &= t \frac{\partial}{\partial x} + xM, \\ D &= x \frac{\partial}{\partial x} + \Delta + 2t \frac{\partial}{\partial t}, \\ K &= t \left(x \frac{\partial}{\partial x} + \Delta + t \frac{\partial}{\partial t} \right) + \frac{1}{2}x^2 M, \end{aligned} \quad (1.195)$$

where Δ is the conformal weight.

⁹ This metric was given first in [333, 485], prior to [543], albeit without relation to the Schrödinger symmetry.

Thus, the vector-field realization of the Schrödinger algebra (1.195) acts on the boundary field $\phi(t, x, x_-)$ with fixed conformal weight Δ .

In this realization the Casimir operator becomes

$$\tilde{C}_4^0 = M^2 C_4^0, \quad C_4^0 = (2\Delta - 1)(2\Delta - 5). \quad (1.196)$$

As expected C_4^0 is a constant which has the same value if we replace Δ by $3 - \Delta$:

$$C_4^0(\Delta) = C_4^0(3 - \Delta). \quad (1.197)$$

This already means that the two boundary fields with conformal weights Δ and $3 - \Delta$ are related, or in mathematical language, that the corresponding representations are (partially) equivalent.

1.3.5 Boundary-to-bulk correspondence

As we explained in the Introduction we concentrate on one aspect of AdS/CFT [293, 572], namely, the holography principle, or boundary-to-bulk correspondence, which means to have an operator which maps a boundary field φ to a bulk field ϕ ; cf. [572, 168].

Mathematically, this means the following. We treat both the boundary fields and the bulk fields as representation spaces of the Schrödinger algebra. The action of the Schrödinger algebra in the boundary, respectively, bulk, representation spaces is given by equations (1.195), respectively, by equations (1.192). The boundary-to-bulk operator maps the boundary representation space to the bulk representation space.

The fields on the boundary are fixed by the value of the conformal weight Δ , correspondingly, as we saw, the Casimir operator has the eigenvalue determined by Δ :

$$\begin{aligned} C_4^0 \varphi(t, x, x_-) &= \lambda \varphi(t, x, x_-), \\ \lambda &= (2\Delta - 1)(2\Delta - 5). \end{aligned} \quad (1.198)$$

Thus, the first requirement for the corresponding field on the bulk $\phi(t, x, x_-, z)$ is to satisfy the same eigenvalue equation, namely, we require

$$\begin{aligned} C_4 \phi(t, x, x_-, z) &= \lambda \phi(t, x, x_-, z), \\ \lambda &= (2\Delta - 1)(2\Delta - 5), \end{aligned} \quad (1.199)$$

where C_4 is the differential operator given in (1.193). Thus, in the bulk the eigenvalue condition is a differential equation.

The other condition is the behaviour of the bulk field when we approach the boundary:

$$\begin{aligned} \phi(t, x, x_-, z) &\rightarrow z^\alpha \varphi(t, x, x_-), \\ \alpha &= \Delta, 3 - \Delta. \end{aligned} \quad (1.200)$$

Let us denote by $\hat{C}^\alpha$ the space of bulk functions $\phi(t, x, x_-, z)$ satisfying (1.199) and (1.200).

To find the boundary-to-bulk operator we first find the two-point Green function in the bulk solving the differential equation:

$$(C_4 - \lambda)G(\chi, z; \chi', z') = z'^4 \delta^3(\chi - \chi') \delta(z - z') \quad (1.201)$$

where $\chi = (t, x_-, x)$.

It is important to use an invariant variable, which here is

$$u = \frac{4zz'}{(x - x')^2 - 2(t - t')(x_- - x'_-) + (z + z')^2}. \quad (1.202)$$

The normalization is chosen so that for coinciding points we have $u = 1$.

In terms of u the Casimir operator becomes

$$C_4 = 4u^2(1 - u) \frac{d^2}{du^2} - 8u \frac{d}{du} + 5. \quad (1.203)$$

The eigenvalue equation can be reduced to the hypergeometric equation by the substitution

$$G(\chi, z; \chi', z') = G(u) = u^\alpha \hat{G}(u) \quad (1.204)$$

and the two solutions are

$$\hat{G}(u) = F(\alpha, \alpha - 1; 2(\alpha - 1); u), \quad \alpha = \Delta, 3 - \Delta, \quad (1.205)$$

where $F = {}_2F_1$ is the standard hypergeometric function.

As expected at $u = 1$ both solutions are singular: by Bateman–Erdelyi [45], they can be recast into

$$\begin{aligned} G(u) &= \frac{u^\Delta}{1 - u} F(\Delta - 2, \Delta - 1; 2(\Delta - 1); u), \quad \alpha = \Delta, \\ G(u) &= \frac{u^{3-\Delta}}{1 - u} F(1 - \Delta, 2 - \Delta; 2(2 - \Delta); u), \quad \alpha = 3 - \Delta. \end{aligned}$$

Now the boundary-to-bulk operator is obtained from the two-point bulk Green function by bringing one of the points to the boundary, however, one has to take into account all information from the field on the boundary.

More precisely, we express the function in the bulk with boundary behaviour (1.200) through the function on the boundary by the formula

$$\phi(\chi, z) = \int d^3\chi' S_\alpha(\chi - \chi', z) \varphi(\chi'), \quad (1.206)$$

where $d^3\chi' = dx'_+ dx'_- dx'$ and $S_\alpha(\chi - \chi', z)$ is defined by

$$S_\alpha(\chi - \chi', z) = \lim_{z' \rightarrow 0} z'^{-\alpha} G(u) = \left[\frac{4z}{(\chi - \chi')^2 - 2(t - t')(\chi_- - \chi'_-) + z^2} \right]^\alpha. \quad (1.207)$$

An important ingredient of this approach is that the bulk-to-boundary and boundary-to-bulk operators are actually intertwining operators. To see this we need some more notation.

Let us denote by L_α the bulk-to-boundary operator:

$$(L_\alpha \phi)(\chi) \doteq \lim_{z \rightarrow 0} z^{-\alpha} \phi(\chi, z), \quad (1.208)$$

where $\alpha = \Delta, 3 - \Delta$ consistently with (1.200). The intertwining property is

$$L_\alpha \circ \hat{X} = \tilde{X}_\alpha \circ L_\alpha, \quad X \in \hat{\mathfrak{s}}(1), \quad (1.209)$$

where $\tilde{X}_\alpha$ denotes the action of the generator X on the boundary (1.195) (with Δ replaced by α from (1.200)), $\hat{X}$ denotes the action of the generator X in the bulk (1.192).

Let us denote by $\tilde{L}_\alpha$ the boundary-to-bulk operator in (1.206):

$$\phi(\chi, z) = (\tilde{L}_\alpha \phi)(\chi, z) \doteq \int d^3\chi' S_\alpha(\chi - \chi', z) \phi(\chi'). \quad (1.210)$$

The intertwining property now is

$$\tilde{L}_\alpha \circ \tilde{X}_{3-\alpha} = \hat{X} \circ \tilde{L}_\alpha, \quad X \in \hat{\mathfrak{s}}(1). \quad (1.211)$$

Next we check consistency of the bulk-to-boundary and boundary-to-bulk operators, namely, their consecutive application in both orders should be the identity map:

$$L_{3-\alpha} \circ \tilde{L}_\alpha = \mathbf{1}_{\text{boundary}}, \quad (1.212)$$

$$\tilde{L}_\alpha \circ L_{3-\alpha} = \mathbf{1}_{\text{bulk}}. \quad (1.213)$$

Checking (1.212) in [6] one obtains

$$(L_{3-\alpha} \circ \tilde{L}_\alpha \phi)(\chi) = 2^{2\alpha} \pi^{3/2} \frac{\Gamma(\alpha - \frac{3}{2})}{\Gamma(\alpha)} \phi(\chi). \quad (1.214)$$

Thus, in order to obtain (1.212) exactly, we have to normalize, e. g. $\tilde{L}_\alpha$.

We note the excluded values $\alpha - 3/2 \notin \mathbb{Z}_+$ for which the two intertwining operators are not inverse to each other. This means that at least one of the representations is reducible. This reducibility was established [175] for the associated Verma modules with lowest weight determined by the conformal weight Δ and is reviewed below.

Checking (1.213) is now straightforward, but it also fails for the excluded values.

Note that checking (1.212) we used (1.208) for $\alpha \rightarrow 3 - \alpha$, i. e. we used one possible limit of the bulk field (1.206). But it is important to note that this bulk field has also the boundary as given in (1.208). Namely, we can consider the field

$$\varphi_0(\chi) \doteq (I_\alpha \phi)(\chi) = \lim_{z \rightarrow 0} z^{-\alpha} \phi(\chi, z), \quad (1.215)$$

where $\phi(\chi, z)$ is given by (1.206). We obtain immediately

$$\varphi_0(\chi) = \int d^3 \chi' G_\alpha(\chi - \chi') \varphi(\chi'), \quad (1.216)$$

where

$$G_\alpha(\chi) = \left[\frac{4}{\chi^2 - 2t\chi_-} \right]^\alpha. \quad (1.217)$$

If we denote by G_α the operator in (1.216) then we have the intertwining property,

$$\tilde{X}_\alpha \circ G_\alpha = G_\alpha \circ \tilde{X}_{3-\alpha}. \quad (1.218)$$

Thus, the two boundary fields corresponding to the two limits of the bulk field are equivalent (partially equivalent for $\alpha \in \mathbb{Z} + 3/2$). The intertwining kernel has the properties of the conformal two-point function.

Thus, for generic Δ the bulk fields obtained for the two values of α are not only equivalent—they coincide, since both have the two fields φ_0 and φ as boundaries.

As in the relativistic case there is a range of dimensions when both fields $\Delta, 3 - \Delta$ are physical:

$$\Delta_-^0 \equiv 1/2 < \Delta < 5/2 \equiv \Delta_+^0. \quad (1.219)$$

At these bounds the Casimir eigenvalue $\lambda = (2\Delta - 1)(2\Delta - 5)$ becomes zero.

1.3.6 Nonrelativistic reduction

In this section we review the connection of [543, 35, 249] with the formalism of [6] reviewed in the previous section. For this purpose, we consider the action for a scalar field in the background (1.191):

$$I(\phi) = - \int d^3 \chi dz \sqrt{-g} (\partial^\mu \phi^* \partial_\mu \phi + m_0^2 |\phi|^2). \quad (1.220)$$

By integrating by parts, and taking into account a nontrivial contribution from the boundary, one can see that $I(\phi)$ obeys the following expression:

$$I(\phi) = \int d^3 \chi dz \sqrt{-g} \phi^* (\partial^\mu \partial_\mu - m_0^2) \phi - \lim_{z \rightarrow 0} \int d^3 \chi \frac{1}{z^3} \phi^* z \partial_z \phi. \quad (1.221)$$

The second term is evaluated using (1.206). For $z \rightarrow 0$, one has

$$z\partial_z\phi \sim \alpha(4z)^\alpha \int d^3\chi' \frac{\varphi(\chi')}{[(x-x')^2 - 2(x_+ - x'_+)(x_- - x'_-)]^\alpha} + O(z^{\alpha+2}). \quad (1.222)$$

It follows that

$$\begin{aligned} \lim_{z \rightarrow 0} \int d^3\chi \frac{1}{z^3} \phi^* z\partial_z\phi &= \lim_{z \rightarrow 0} \alpha \int d^3\chi d^3\chi' z^{\alpha-3} \phi^*(\chi, z) \left(\frac{4}{A}\right)^\alpha \varphi(\chi') \\ &= 4^\alpha \alpha \int d^3\chi d^3\chi' \frac{\varphi(\chi)^* \varphi(\chi')}{[(x-x')^2 - 2(x_+ - x'_+)(x_- - x'_-)]^\alpha}. \end{aligned} \quad (1.223)$$

The equation of motion being read off from the first term in (1.221) can be expressed in terms of the differential operator (1.193):

$$(\partial^\mu \partial_\mu - m_0^2)\phi = \left(\frac{C_4 - 5}{4} + 2\partial_-^2 - m_0^2\right)\phi = 0. \quad (1.224)$$

The fields in the bulk (1.206) do not solve the equation of motion. Now we set an Ansatz for the fields on the boundary: $\varphi(\chi) = e^{Mx_-} \varphi(x_+, x)$. Further we compactify the x_- coordinate: $x_- + a \sim x_-$ as in, e. g. [278, 39]. This leads to a separation of variables for the fields in the bulk in the following way:

$$\phi(\chi, z) = e^{Mx_-} \int dx'_+ dx' \int_0^a d\xi \left(\frac{4z}{(x-x')^2 - 2(x_+ - x'_+)\xi + z^2} \right)^\alpha e^{-M\xi} \varphi(x'_+, x').$$

Thus we are allowed to make the identification $\partial_{x_-} = M$ both in the bulk and on the boundary [543, 35]. We remark that under this identification the operator (1.194) becomes the Schrödinger operator. Integration over ξ turns out to be incomplete gamma function:

$$\phi(\chi, z) = e^{Mx_-} \phi(x_+, x, z), \quad (1.225)$$

$$\begin{aligned} \phi(x_+, x, z) &= (-2z)^\alpha M^{\alpha-1} \gamma(1-\alpha, Ma) \\ &\times \int \frac{dx'_+ dx'}{(x_+ - x'_+)^{\alpha-1}} \exp\left(-\frac{(x-x')^2 + z^2}{2(x_+ - x'_+)} M\right) \varphi(x'_+, x'). \end{aligned} \quad (1.226)$$

This formula was obtained first in [249]. The equation of motion (1.224) now reads

$$\left(\frac{\lambda - 5}{4} - m^2\right)\phi(x_+, x, z) = 0, \quad (1.227)$$

where $m^2 = m_0^2 - 2M^2$. Requiring $\phi(x_+, x, z)$ to be a solution to the equation of motion makes the connection between the conformal weight and mass:

$$\Delta_\pm = \frac{1}{2}(3 \pm \sqrt{9 + 4m^2}). \quad (1.228)$$

This result is identical to the relativistic AdS/CFT correspondence [293, 572]. The action (1.221) evaluated for these classical solutions has the following form ($\alpha = \Delta_{\pm}$):

$$I(\phi) = -(-2)^\alpha M^{\alpha-1} \alpha \gamma (1 - \alpha, Ma) \\ \times \int \frac{dx dx_+ dx'_+}{(x_+ - x'_+)^{\alpha}} \exp\left(-\frac{(x - x')^2}{2(x_+ - x'_+)} M\right) \varphi(x_+, x)^* \varphi(x'_+, x'). \quad (1.229)$$

The two-point function of the operator dual to ϕ computed from (1.229) coincides with the result of [543, 35, 308, 547]. We remark that the Ansatz for the boundary fields $\varphi(\chi) = \exp(Mx_- - \omega x_+ + ikx)$ used in [543, 35] is not necessary to derive (1.229).

One can also recover the solutions in [543, 35] rather simply in the group-theoretical context of [6]. We use again the eigenvalue problem of the differential operator (1.193):

$$C_4 \phi(x_+, x, z) = \lambda \phi(x_+, x, z), \quad (1.230)$$

but we make a separation of variables $\phi(x_+, x, z) = \psi(x_+, x) f(z)$. Then (1.230) is written as follows:

$$\frac{1}{f(z)} \left(\frac{\partial^2}{\partial z^2} - \frac{2}{z} \partial_z + \frac{5 - \lambda}{4z^2} \right) f(z) = \frac{1}{\psi(x_+, x)} \hat{S} \psi(x_+, x) = p^2 \text{ (const)}.$$

The Schrödinger part is easily solved: $\psi(x_+, x) = \exp(-\omega x_+ + ikx)$ which gives

$$p^2 = -2M\omega + k^2. \quad (1.231)$$

The equation for $f(z)$ now becomes

$$\frac{\partial^2}{\partial z^2} f(z) - \frac{2}{z} \partial_z f(z) + \left(2M\omega - k^2 - \frac{m^2}{z^2} \right) f(z) = 0. \quad (1.232)$$

This is the equation given in [543, 35] for $d = 1$. Thus solutions to equation (1.232) are given by modified Bessel functions: $f_{\pm}(z) = z^{3/2} K_{\pm\nu}(pz)$ where ν is related to the effective mass m [543, 35]. In the group-theoretic approach one can see its relation to the eigenvalue of C_4 : $\nu = \sqrt{\lambda + 4}/2$ [6].

We close this section by giving the expression of (1.229) for the alternate boundary field φ_0 . To this end, we again use the Ansatz $\varphi(\chi) = e^{Mx_-} \varphi(x_+, x)$ for (1.216). Then performing the integration over x'_- it is immediate to see that:

$$\varphi_0(x, x_+) \sim e^{Mx_-} \int \frac{dx' dx'_+}{(x_+ - x'_+)^{\alpha}} \exp\left(-\frac{(x - x')^2}{2(x_+ - x'_+)} M\right) \varphi(x'_+, x'). \quad (1.233)$$

One can invert this relation since $G_{3-\alpha} \circ G_{\alpha} = 1_{\text{boundary}}$. Substitution of (1.233) and its inverse to (1.229) gives the following expression:

$$I(\phi) \sim \int \frac{dx dx_+ dx'_+}{(x_+ - x'_+)^{3-\alpha}} \exp\left(-\frac{(x - x')^2}{2(x_+ - x'_+)} M\right) \varphi_0(x_+, x)^* \varphi_0(x'_+, x'). \quad (1.234)$$

2 Non-relativistic invariant differential operators and equations

Summary

We give a review of some group-theoretical results related to non-relativistic holography. Our main playgrounds are the Schrödinger equation and the Schrödinger algebra. We recall the fact that there is a hierarchy of equations on the boundary, invariant w. r. t. the Schrödinger algebra. The derivation of this hierarchy uses a mechanism introduced first for semisimple Lie groups and adapted to the non-semisimple Schrödinger algebra. This requires development of the representation theory of the Schrödinger algebra, which is reviewed in some detail.

In Section 2.1 the Schrödinger equation is reviewed as an invariant differential equation in the $(1 + 1)$ -dimensional case. On the boundary this was done in [175] (extending the approach in the semisimple group setting [161]), constructing actually an infinite hierarchy of invariant differential equations, the first member being the free heat/Schrödinger equation). In Section 2.1.4 the extension of this construction is reviewed to the bulk combining techniques from [6] and [175]. In Section 2.2 the Schrödinger equation is reviewed as an invariant differential equation in the general $(n + 1)$ -dimensional case following [8, 190]. The general situation is very complicated and requires separate study of the cases $n = 2N$ and $n = 2N + 1$. In Section 2.3 the $(3 + 1)$ -dimensional case is reviewed separately and in more detail, since it is most important for physical applications. In Section 2.4 the q -deformation is reviewed of the Schrödinger algebra in the $(1 + 1)$ -dimensional case; cf. [176]. In Section 2.5 the difference analogues of the Schrödinger algebra in the $(n + 1)$ -dimensional case are reviewed; cf. [177].

2.1 Non-relativistic invariant differential equations for $\hat{\mathcal{S}}(1)$

2.1.1 Canonical procedure

In this subsection, we briefly outline the method of [161] that will be used subsequently. Let G be a complex semisimple Lie group and $\mathfrak{g}$ its Lie algebra. Let $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_-$ be the standard triangular decomposition. We consider representations of $\mathfrak{g}$ whose representation spaces $\mathcal{C}_\Lambda$ are $\mathbb{C}^\infty$ functions $\mathcal{F}$ on G with the property called *right covariance*

$$\mathcal{F}(g x g_-) = e^{\Lambda(H)} \mathcal{F}(g), \quad g \in G, x = e^H \in G_0, H \in \mathfrak{g}_0, g_- \in G_-, \quad (2.1)$$

where $\Lambda \in \mathfrak{g}_0^*$, $\Lambda(H) \in \mathbb{Z}$, $G_0 \equiv \exp \mathfrak{g}_0$, $G_\pm \equiv \exp \mathfrak{g}_\pm$. Thus the functions of $\mathcal{C}_\Lambda$ are actually functions on the coset G/B , $B \equiv G_0 G_-$ being a Borel subgroup, or equivalently [161], on G_+ , which is dense in G/B . We denote by $\mathcal{C}_\Lambda$ the restricted representation spaces of functions f on G_+ , such that $f = \mathcal{F}|_{G_+}$. We introduce the left $\pi_L(X)$ and right $\pi_R(X)$ actions of $\mathfrak{g}$ on $\mathcal{C}_\Lambda$ by the standard formulae

$$\pi_L(X) \mathcal{F}(g) \equiv \left. \frac{d}{d\tau} \mathcal{F}(e^{-\tau X} g) \right|_{\tau=0}, \quad \pi_R(X) \mathcal{F}(g) \equiv \left. \frac{d}{d\tau} \mathcal{F}(g e^{\tau X}) \right|_{\tau=0}, \quad (2.2)$$

where $X \in \mathfrak{g}$. The left action gives representations of $\mathfrak{g}$ by differential operators. It may be considered for arbitrary $\Lambda(H) \in \mathbb{C}$ and it may be restricted to C_Λ . On the other hand, the space C_Λ has a lowest weight structure with respect to the right action, since one can show, using the right covariance, that

$$\pi_R(H)\mathcal{F}(g) = \Lambda(H)\mathcal{F}(g), \quad \pi_R(X)\mathcal{F}(g) = 0, \quad (2.3)$$

where $H \in \mathfrak{g}_0$ and $X \in \mathfrak{g}_-$. Thus we are prompted to employ properties of the Verma module V^Λ with the lowest weight Λ , such that $V^\Lambda \simeq U(\mathfrak{g}_+)\nu_0$, where ν_0 is the lowest weight vector, $U(\mathfrak{g}_+)$ is the universal enveloping algebra of $\mathfrak{g}_+$. When a Verma module is reducible, it has (at least one) *singular vector* ν_s such that $H\nu_s = \Lambda'(H)\nu_s$, $\Lambda' \neq \Lambda$, $H \in \mathfrak{g}_0$, $X\nu_s = 0$, $X \in \mathfrak{g}_-$. It has the general structure $\nu_s = P(\mathfrak{g}_+)\nu_0$, where $P(\mathfrak{g}_+)$ is a polynomial of the generators of $\mathfrak{g}_+$. Then it is shown that the same polynomial $P(\mathfrak{g}_+)$ gives rise to a $\mathfrak{g}$ -invariant differential equation given explicitly by $P(\pi_R(\mathfrak{g}_+))\psi = 0$, $\psi = \mathcal{F}, f$.

Below the procedure of [161] shall be used in our non-semisimple Schrödinger setting.

2.1.2 Verma modules and singular vectors

In this subsection we follow [175]; see also [8, 190]. We consider lowest weight modules (LWMs) over $\hat{\mathfrak{s}}(n)$, in particular, Verma modules, which are standard for semisimple Lie algebras (SSLAs) and their q -deformations. (For more information on representations of the Schrödinger algebra we refer to [510, 219, 61, 95, 557, 547, 31, 4, 9, 577, 21, 218].)

A lowest weight module (LWM) M^Λ over $\hat{\mathfrak{s}}(n)$ is given by the lowest weight $\Lambda \in \mathcal{H}^*$ ($\mathcal{H}^*$ is the dual of $\hat{\mathfrak{s}}(n)^0$) and a lowest weight vector ν_0 so that $X\nu_0 = 0$ if $X \in \hat{\mathfrak{s}}(n)^-$, $H\nu_0 = \Lambda(H)\nu_0$ if $H \in \hat{\mathfrak{s}}(n)^0$. In particular, we use the Verma modules V^Λ over $\hat{\mathfrak{s}}(n)$ which are the lowest weight modules induced from a one-dimensional representation of the analogue of a Borel subalgebra $\mathcal{B}^0 \oplus \hat{\mathfrak{s}}(n)^-$ spanned by ν_0 . The Verma module is given explicitly by $V^\Lambda \cong U(\hat{\mathfrak{s}}(n)^+) \otimes \nu_0$, where $U(\hat{\mathfrak{s}}(n)^+)$ is the universal enveloping algebra of $\hat{\mathfrak{s}}(n)^+$. Further, for brevity we shall omit the sign $\otimes$, i. e. we shall write instead of $\otimes \nu_0$ just ν_0 .

Now we restrict to the case $n = 1$. Then the Cartan subalgebra $\hat{\mathfrak{s}}(1)^0$ is generated by D, M and we can write all above-mentioned properties as

$$\begin{aligned} D\nu_0 &= \Delta\nu_0, & M\nu_0 &= M\nu_0, \\ P_x\nu_0 &= 0, & P_t\nu_0 &= 0, \end{aligned} \quad (2.4)$$

where $\Delta \in \mathbb{R}$ is the (conformal) weight.

The Borel subalgebra $\mathcal{B}$ is generated by the nonpositively graded generators D, M, P_x, P_t . Now we denote the Verma module as V^Δ since M is constant.

Clearly, $U(\hat{S}^+)$ is abelian and has basis elements $p_{k,\ell} = G^k K^\ell$. The basis vectors of the Verma module are $v_{k,\ell} = p_{k,\ell} \otimes v_0$, (with $v_{0,0} = v_0$). The action of $\hat{S}$ on this basis is derived easily from (1.189):

$$\begin{aligned} Dv_{k,\ell} &= (k + 2\ell + \Delta)v_{k,\ell}, \\ Gv_{k,\ell} &= v_{k+1,\ell}, \\ Kv_{k,\ell} &= v_{k,\ell+1}, \\ P_x v_{k,\ell} &= \ell v_{k+1,\ell-1} + Mkv_{k-1,\ell}, \\ P_t v_{k,\ell} &= \ell(k + \ell - 1 + \Delta)v_{k,\ell-1} + M \frac{k(k-1)}{2} v_{k-2,\ell}. \end{aligned} \quad (2.5)$$

Because of (2.5) we notice that the Verma module V^Δ can be decomposed in homogeneous (w. r. t. D) subspaces as follows:

$$\begin{aligned} V^\Delta &= \oplus_{n=0}^{\infty} V_n^\Delta, \\ V_n^\Delta &= \text{lin.span.}\{v_{k,\ell} \mid k + 2\ell = n\}, \\ \dim V_n^\Delta &= 1 + \left\lfloor \frac{n}{2} \right\rfloor. \end{aligned} \quad (2.6)$$

Next we analyze the reducibility of V^Δ through the singular vectors. In analogy to the SSLA situation (cf., e. g. [161]) a singular vector v_s here is a homogeneous element of V^Δ , such that $v_s \notin \mathbb{C}v_0$, and

$$P_x v_s = 0, \quad P_t v_s = 0. \quad (2.7)$$

All possible singular vectors were given explicitly in [175], where the following was proved.

Proposition 1. *The singular vectors of the Verma module V^Δ over $\hat{S}$ are given as follows:*

$$\begin{aligned} v_s^p &= a_0 \sum_{\ell=0}^{p/2} (-2M)^\ell \binom{p/2}{\ell} v_{p-2\ell,\ell} \\ &= a_0 (G^2 - 2MK)^{p/2} \otimes v_0, \quad \Delta = \frac{3-p}{2}, \quad p \in 2\mathbb{N}, Ma_0 \neq 0, \\ v_s^p &= a_0 v_{p0} = a_0 G^p \otimes v_0, \quad \Delta \text{ arbitrary}, \quad p \in \mathbb{N}, M = 0, a_0 \neq 0. \end{aligned} \quad (2.8) \quad \diamond$$

Remark 1. We stress the very different character of the representations for $M \neq 0$ and $M = 0$ from one another and furthermore from the semisimple case. For $M \neq 0$ and fixed lowest weight at most one singular vector may exist and that vector can be only of *even* grade. For $M = 0$ an infinite number of singular vectors exist – one for each positive grade—and there is no restriction on the weight. This difference is because the value $M = 0$ changes the algebra—it is not a centrally extended one anymore. Both cases *differ from the semisimple case*. To compare we take, e. g. the algebra $\mathfrak{sl}(2)$

since it also has only one Cartan generator as $\hat{s}(1)$. For $\mathfrak{sl}(2)$ for a fixed lowest weight only one singular vector is possible, however, for any 'grade' $n\beta$, where $n \in \mathbb{N}$, β the positive root of $\mathfrak{sl}(2)$, and not just for even n . $\diamond$

Whenever there is a singular vector, the Verma module is reducible. We could analyze this reducibility also via an analogue of the Shapovalov form [539] used in the semisimple case. This is a bilinear form which we define using the involutive antiautomorphism of the Schrödinger algebra (cf. (1.179)):

$$\omega(P_t) = K, \quad \omega(P_x) = G, \quad \omega(D) = D, \quad \omega(M) = M. \quad (2.9)$$

Explicitly, the form here is given by

$$\begin{aligned} (v_{k\ell}, v_{k'\ell'}) &= (p_{k\ell} \otimes v_0, p_{k'\ell'} \otimes v_0) \equiv (v_0, \omega(p_{k\ell})p_{k'\ell'} \otimes v_0) \\ &= (v_0, P_t^\ell P_x^{k'} G^{k'} K^{\ell'} v_0) \end{aligned} \quad (2.10)$$

supplemented by the normalization condition $(v_0, v_0) = 1$. Clearly, subspaces with different weights are orthogonal w. r. t. to this form:

$$(v_{k\ell}, v_{k'\ell'}) \sim \delta_{k+2\ell, k'+2\ell'}. \quad (2.11)$$

To show this for $k+2\ell > k'+2\ell'$ we move all P_t and P_x operators to the right until there are no G and K operators left, while for $k+2\ell < k'+2\ell'$ we first rewrite the LHS of (2.11) as

$$\begin{aligned} (v_{k\ell}, v_{k'\ell'}) &= (p_{k\ell} \otimes v_0, p_{k'\ell'} \otimes v_0) = (\omega(p_{k'\ell'})p_{k\ell} \otimes v_0, v_0) \\ &= (P_t^{\ell'} P_x^{k'} G^k K^\ell v_0, v_0) \end{aligned} \quad (2.12)$$

and then again move all P_t and P_x operators to the right until there are no G and K operators left. The above also shows that the form is symmetric. In the case $k+2\ell = k'+2\ell'$ we have the following explicit expression:

$$\begin{aligned} (v_{k,\ell}, v_{k+2a,\ell-a}) &= \frac{m^{k+a} k! \ell! (k-d+a)_{\ell-a} (k+1)_{2a}}{2^a a!} \\ &\quad \times {}_3F_2\left(\frac{k}{2}, \frac{1-k}{2}, a-\ell; 1+a, 1-k+d-\ell; 1\right) \end{aligned} \quad (2.13)$$

where $a \in \mathbb{Z}_+$, $(a)_p \doteq \Gamma(a+p)/\Gamma(a)$ is the Pochhammer symbol and ${}_3F_2(a, b, c; a', b'; y)$ is a generalized hypergeometric series:

$${}_3F_2(a, b, c; a', b'; y) \doteq \sum_{s \in \mathbb{Z}_+} \frac{(a)_s (b)_s (c)_s}{s! (a')_s (b')_s} y^s, \quad (2.14)$$

which for a , or b , or $c \in \mathbb{Z}_-$ reduces to a polynomial.

A singular vector is orthogonal to any other vector w. r. t. to the form (2.10). Thus we expect (as in the SSLA case) to obtain the same reducibility results analyzing the so-called determinant formula. The determinant formula is the determinant of the matrix $\mathcal{M}_p$ of all Shapovalov forms at a fixed grade p . In [175] was made the following conjecture for $\det \mathcal{M}_p$:

$$\det \mathcal{M}_p = \text{const.} (p) m^{\alpha_p} \prod_{i=0}^{\lfloor \frac{p}{2} \rfloor - 1} (2\Delta - 1 + 2i)^{\lfloor \frac{p}{2} \rfloor - i}, \quad (2.15)$$

$$\alpha_p = \begin{cases} \frac{p^2+2p}{4} & \text{for } p \text{ even,} \\ \frac{(p+1)^2}{4} & \text{for } p \text{ odd,} \end{cases}$$

which there was verified for $p \leq 6$. We see that the determinant has zeroes exactly in the cases when we have singular vectors. The above conjecture was proved in [10].

Further we consider the consequences of the reducibility of the Verma modules. We start with $M \neq 0$ and consider the subspace of $V^{(3-p)/2}$:

$$I^{(3-p)/2} = U(S^+) v_s^p. \quad (2.16)$$

It is invariant under the action of the Schrödinger algebra. Indeed, all vectors have grade $\geq \deg_{\min} = p$: lower grades cannot be achieved since the negative grade generators annihilate v_s^p . Furthermore this subspace is isomorphic to a Verma module $V^{d'}$ with shifted weight $\Delta' = \Delta + p = (p+3)/2$. The latter Verma module has no singular vectors, since its weight is restricted from below: $\Delta' \geq 5/2$, while by (2.8) the necessary weight is $\leq 1/2$.

Let us denote the factor-module $V^{(3-p)/2}/I^{(3-p)/2}$ by $\mathcal{L}^{(3-p)/2}$. Let us denote by $|p\rangle$ the lowest weight vector of $\mathcal{L}^{(3-p)/2}$. It satisfies the following conditions:

$$P_x |p\rangle = 0, \quad (2.17a)$$

$$P_t |p\rangle = 0, \quad (2.17b)$$

$$(G^2 - 2MK)^{p/2} |p\rangle = 0. \quad (2.17c)$$

Consider in more detail the simplest example $p = 2$. In this case the last condition (2.17c) is

$$G^2 |2\rangle = 2MK |2\rangle, \quad (2.18)$$

i. e. K can be replaced by $G^2/2M$. Thus, all vectors of a fixed grade are proportional:

$$G^k K^\ell |2\rangle = \frac{1}{(2M)^\ell} G^{k+2\ell} |2\rangle, \quad \Delta = 1/2, \quad (2.19)$$

so all graded subspaces are one-dimensional, i. e. we have a *singleton basis*:

$$\dim V_n^{1/2} = 1, \quad \forall n, \quad (2.20)$$

which is given only in terms of G . (The term ‘singleton’ was used first by Flato–Fronsdal [231] for two special representations of the algebra $\mathfrak{so}(3, 2)$ discovered by Dirac (for more information, see also Volume 1).)

Analogously, for arbitrary $p \in 2\mathbb{N}$ and $\Delta = (3 - p)/2$, from (2.17c) we see that

$$K^{p/2}|p\rangle = - \sum_{\ell=0}^{p/2-1} \frac{1}{(-2M)^{p/2-\ell}} \binom{p/2}{\ell} G^{p-2\ell} K^\ell |p\rangle. \quad (2.21)$$

Applying repeatedly this relation to the basis one can get rid of all powers of K which are $\geq p/2$. Thus the basis of $\mathcal{L}^{(3-p)/2}$ will be quasi-singleton if $p \geq 4$, namely,

$$\dim V_n^{(3-p)/2} = 1, \quad \text{for } n = 0, 1 \text{ or } n \geq p, \quad (2.22)$$

and it is given by

$$v_{k\ell}^p \equiv G^k K^\ell |p\rangle, \quad p \in 2\mathbb{N}, \quad k, \ell \in \mathbb{Z}_+, \quad \ell \leq p/2 - 1, \quad \Delta = \frac{3-p}{2}. \quad (2.23)$$

The transformation of this basis is easily obtained from (2.5):

$$\begin{aligned} Dv_{k,\ell}^p &= \left(k + 2\ell + \frac{3-p}{2} \right) v_{k,\ell}^p, \\ Gv_{k,\ell}^p &= v_{k+1,\ell}^p, \\ Kv_{k,\ell}^p &= \begin{cases} v_{k,\ell+1}^p & \ell < \frac{p}{2} - 1, \\ - \sum_{s=0}^{p/2-1} \frac{1}{(-2M)^{p/2-s}} \binom{p/2}{s} v_{k+p-2s,s}^p & \ell = \frac{p}{2} - 1, \end{cases} \\ P_x v_{k,\ell}^p &= \ell v_{k+1,\ell-1}^p + mk v_{k-1,\ell}^p, \\ P_t v_{k,\ell}^p &= \ell \left(k + \ell + \frac{1-p}{2} \right) v_{k,\ell-1}^p + m \frac{k(k-1)}{2} v_{k-2,\ell}^p. \end{aligned} \quad (2.24)$$

From the transformation rules we see that $\mathcal{L}^{(3-p)/2}$ is irreducible. It is also clear that in the simplest case $p = 2$ the irrep $\mathcal{L}^{1/2}$ is also an irrep of the centrally extended Galilean subalgebra $\hat{\mathcal{G}}(1)$ spanned by P_x, P_t, G .

For $M = 0$ we consider the subspaces of V^Δ :

$$I_p^\Delta = U(S^+) G^p \otimes v_0, \quad p \in \mathbb{N}. \quad (2.25)$$

They are invariant under the action of the Schrödinger algebra, which is shown as in the case $M \neq 0$. The corresponding singular vectors are

$$\tilde{v}_s^p = G^p \otimes v_0. \quad (2.26)$$

Furthermore the subspace I_p^Δ is isomorphic to a Verma module $V^{\Delta'}$ with shifted weight $\Delta' = \Delta + p$. The latter Verma module again has an infinite number of singular vectors,

and an infinite number of subspaces $I_{p'}^{\Delta+p} = U(S^+)G^{p'} \otimes v_0$, $p' \in \mathbb{N}$, isomorphic to Verma modules $V^{\Delta+p+p'}$. Furthermore, the original Verma module V^Δ is itself a submodule of an infinite number of Verma modules $V^{\Delta+p''}$, $p'' \in \mathbb{N}$. Altogether, for each Δ there exists a doubly infinite sequence of Verma modules:

$$\dots \supset V^{\Delta-1} \supset V^\Delta \supset V^{\Delta+1} \supset \dots, \quad M = 0, \Delta \text{ arbitrary.} \quad (2.27)$$

Of course, all V^Δ whose weights differ by an integer are in one and the same sequence. Such embedding diagrams were called multiplets in [153] (see also the earlier volumes).

For each V^Δ the submodule $I_1^\Delta \cong V^{\Delta+1}$ contains as submodules all other submodules of V^Δ , i. e. in (2.26) only the singular vector with $p = 1$ is relevant. Consider the factor space $\tilde{\mathcal{L}}_0^\Delta = V^\Delta / V^{\Delta+1}$ and denote by $\widetilde{|0\rangle}$ its lowest weight vector. The latter satisfies the following conditions:

$$P_x \widetilde{|0\rangle} = 0, \quad (2.28a)$$

$$P_t \widetilde{|0\rangle} = 0, \quad (2.28b)$$

$$G \widetilde{|0\rangle} = 0. \quad (2.28c)$$

Consequently, the basis of $\tilde{\mathcal{L}}_0^d$ is given by

$$\hat{\varphi}_\ell^0 = K^\ell \widetilde{|0\rangle}. \quad (2.29)$$

This is another example of an even smaller than singleton basis, since the odd-graded levels are empty. Thus, we call the basis in (2.29) a *singleton-void basis*. Its transformation rules are (cf. (2.5))

$$D\hat{\varphi}_\ell^0 = (2\ell + \Delta)\hat{\varphi}_\ell^0, \quad (2.30a)$$

$$G\hat{\varphi}_\ell^0 = 0, \quad (2.30b)$$

$$K\hat{\varphi}_\ell^0 = \hat{\varphi}_{\ell+1}^0, \quad (2.30c)$$

$$P_x \hat{\varphi}_\ell^0 = 0, \quad (2.30d)$$

$$P_t \hat{\varphi}_\ell^0 = \ell(\ell - 1 + \Delta)\hat{\varphi}_{\ell-1}^0. \quad (2.30e)$$

Clearly, $\tilde{\mathcal{L}}_0^\Delta$ is in fact a Verma module over the $\mathfrak{sl}(2, \mathbb{R})$ subalgebra spanned by D, K, P_t . It is well known that such a Verma module is reducible (cf., e. g. [161]) iff $\Delta \in \mathbb{Z}_-$. In this case there exists a singular vector given by

$$v_s^0 = \hat{\varphi}_{1-\Delta}^0 = K^{1-\Delta} \widetilde{|0\rangle}, \quad \Delta \in \mathbb{Z}_-. \quad (2.31)$$

Thus the invariant subspace $\tilde{I}_0^\Delta$ of $\tilde{\mathcal{L}}_0^\Delta$ is spanned by $K^{1-\Delta+\ell} \widetilde{|0\rangle}$, $\ell \in \mathbb{Z}_+$, and is isomorphic to another Verma module $\tilde{\mathcal{L}}_0^{\Delta-2}$ which is irreducible.

Let us stress that v_s^0 is not a singular vector of the original Verma module V^Δ , but of its factor module $\tilde{\mathcal{L}}_0^\Delta$. Vectors which become singular vectors only in factor modules are called *subsingular vectors* [167].

Consider now the factor space $\mathcal{L}_0^\Delta = \tilde{\mathcal{L}}_0^\Delta / \tilde{\mathcal{L}}_0^{\Delta-2}$, denoting by $|0\rangle$ its lowest weight vector. It satisfies the following conditions:

$$P_x|0\rangle = 0, \quad (2.32a)$$

$$P_t|0\rangle = 0, \quad (2.32b)$$

$$G|0\rangle = 0, \quad (2.32c)$$

$$K^{1-\Delta}|0\rangle = 0. \quad (2.32d)$$

Consequently, the basis of $\mathcal{L}_0^\Delta$ is given by

$$v_\ell^0 = K^\ell|0\rangle, \quad -\Delta, \ell \in \mathbb{Z}_+, \ell \leq -\Delta. \quad (2.33)$$

Hence $\mathcal{L}_0^\Delta$ is finite-dimensional: $\dim \mathcal{L}_0^\Delta = 1 - \Delta$, and in fact, when Δ runs through $\mathbb{Z}_-$ one obtains all irreducible finite-dimensional representations of $\mathfrak{sl}(2, \mathbb{R})$. The latter are not unitary, except in the trivial one-dimensional case obtained for $\Delta = 0$. The transformation rules for v_ℓ^0 are as for $\hat{\phi}_\ell^0$ in (2.30), except that $Kv_\Delta^0 = 0$.

Summarizing the above in [175] the following was proved.

Theorem 1. *The list of the irreducible lowest weight modules over the (centrally extended) Schrödinger algebra is given by*

- V^d , when $\Delta \neq (3-p)/2$, $p \in 2\mathbb{N}$ and $M \neq 0$;
- $\mathcal{L}^{(3-p)/2}$, when $\Delta = (3-p)/2$, $p \in 2\mathbb{N}$ and $M \neq 0$;
- $\tilde{\mathcal{L}}_0^\Delta$, when $\Delta \notin \mathbb{Z}_-$ and $M = 0$;
- $\mathcal{L}_0^\Delta$, when $d \in \mathbb{Z}_-$ and $M = 0$.

In the last case one has $\dim \mathcal{L}_0^\Delta = 1 - \Delta$; in all other cases the irreps are infinite-dimensional. The representation $\mathcal{L}^{1/2}$ is also an irrep of the centrally extended Galilean subalgebra $\hat{\mathcal{G}}(1)$. The irreps in the last two cases are also irreps of the subalgebra $\mathfrak{sl}(2, \mathbb{R})$.

2.1.3 Generalized Schrödinger equations from a vector-field realization of the Schrödinger algebra

Now we shall employ the vector-field representation (1.195) as in [175]. This realization was used to construct a polynomial realization of the irreducible lowest weight modules considered in the previous subsection. For this realization we represent the lowest weight vector by the function 1. Indeed, the constants in (1.195) are chosen so that (2.4) is satisfied:

$$D1 = \Delta, \quad M1 = M, \quad P_x 1 = 0, \quad P_t 1 = 0. \quad (2.34)$$

Applying the basis elements $p_{k,\ell} = G^k K^\ell$ of the universal enveloping algebra $U(\hat{S}^+)$ to 1 we get polynomials in x, t . Let us denote these polynomials by $f_{k,\ell} \equiv p_{k,\ell} 1$. (In partial cases we have explicit expressions for $f_{k,\ell}$ from [175] but we shall not need them here.)

Let us denote by C^Δ the spaces spanned by the elements $f_{k,\ell}$, and by L^Δ the irreducible subspace of C^Δ . Now in [175] the following was shown.

Theorem 2. *The irreducible spaces L^Δ give a realization of the irreducible lowest weight representations of $\hat{S}(1)$ given in Theorem 1.* $\diamond$

We consider now in more detail the most interesting cases of the representations $L^{(3-p)/2}$ with $M \neq 0$ and $p \in 2\mathbb{N}$.

We first introduce an operator by the polynomial $G^2 - 2MK \in U(\hat{S}^+)$ expressing this polynomial in the vector-field realization:

$$S \doteq G^2 - 2MK = t^2(\partial_x^2 - 2M\partial_t) + 2Mt\left(\frac{1}{2} - \Delta\right). \quad (2.35)$$

In these case we have the following [175].

Proposition 2. *Each basis polynomial $f_{k,\ell}$ of $L^{(3-p)/2}$ satisfies*

$$\begin{aligned} S^{p/2} f_{k,\ell} &= (t^2(\partial_x^2 - 2M\partial_t) + (p-2)Mt)^{p/2} f_{k,\ell} \\ &= t^p(\partial_x^2 - 2M\partial_t)^{p/2} f_{k,\ell} = 0, \quad \Delta = \frac{3-p}{2}. \end{aligned} \quad (2.36) \quad \diamond$$

Thus we have obtained in (2.36) an infinite hierarchy of PDOs $S^{p/2}$ which give rise to differential equations, we call **free generalized heat/Schrödinger equations**. The equations are obtained by substituting the vector-field realization in the singular vectors (thus extending the procedure of [161]). This substitution gives

$$\begin{aligned} S^{p/2} &= (G^2 - 2MK)^{p/2} = (t^2(\partial_x^2 - 2M\partial_t) + (p-2)mt)^{p/2} \\ &= t^p(\partial_x^2 - 2M\partial_t)^{p/2}. \end{aligned} \quad (2.37)$$

Thus, the hierarchy of equations is

$$t^p(\partial_x^2 - 2M\partial_t)^{p/2} f = 0. \quad (2.38)$$

In the case of function spaces with elements which are polynomials in t (as our representation spaces) or are singular at most as $t^{-p/2}$ for $t \rightarrow 0$, the hierarchy is

$$(\partial_x^2 - 2M\partial_t)^{p/2} f = 0. \quad (2.39)$$

The above proposition also shows that the representation spaces are comprised from solutions of the corresponding equations (2.39). The case $p = 2$ and M real is the ordinary heat or diffusion equation and for $p = 2$ and M purely imaginary we get the

free Schrödinger equation. So the members of the hierarchy of equations which are invariant under the Schrödinger group have generically higher orders of derivatives in t . This shows that the Schrödinger symmetry is not necessarily connected with first order (in t) differential operators.

We can further extend [161] to the non-semisimple situation by considering equations with nonzero RHS. However, invariance w.r.t. the Schrödinger algebra requires that the RHS is an element of the irreducible representation space $C^{(p+3)/2}$, while the functions in the LHS are not restricted to the solution subspace of (2.39). Thus, using the operator in (2.38) we obtained the following hierarchy of generalized heat/Schrödinger equations:

$$t^p(\partial_x^2 - 2M\partial_t)^{p/2}f = j, \quad f \in C^{(3-p)/2}, \quad j \in C^{(3+p)/2}. \quad (2.40)$$

Remark 2. It is interesting to note that (2.40) looks similar to an hierarchy of equations involving the d'Alembert operator and conditionally invariant w.r.t. conformal algebra $\mathfrak{su}(2, 2)$:

$$\square^n \hat{\varphi}(\mathbf{x}) = \hat{\varphi}'(\mathbf{x}), \quad n \in \mathbb{N}, \quad (2.41)$$

where $\hat{\varphi}, \hat{\varphi}'$ are scalar fields of different fixed conformal weights depending on n , $\mathbf{x} = (x_0, x_1, x_2, x_3)$ denotes the Minkowski space-time coordinates, and $\square$ is the d'Alembert operator: $\square = \partial^\mu \partial_\mu = (\vec{\partial})^2 - (\partial_0)^2$; see, e. g. [167]. $\diamond$

Of course, we may consider more general function spaces of the variables t, x , say $C^\infty(\mathbb{R}^2, \mathbb{R})$, on which the centrally extended Schrödinger algebra is acting by formulae (1.195). For the example of the ordinary heat equation one may find also solutions of the type

$$(\alpha \cosh(\sqrt{\lambda}x) + \beta \sinh(\sqrt{\lambda}x)) \exp\left(\frac{\lambda t}{2M}\right), \quad \alpha, \beta, \lambda \in \mathbb{R}, \quad (2.42)$$

which results from separation of the t and x variables, while the polynomial solutions above may be obtained also by separation of the variables t and $Mx^2/2t$; cf. [175].

2.1.4 Generalized Schrödinger equations in the bulk

In this subsection we review [170]. Now we shall employ the bulk vector-field representation (1.192) trying similarly to the previous subsection to construct generalized Schrödinger equations in the bulk. We start with the operator (distinguishing bulk operators by hats)

$$\hat{S} \doteq \hat{G}^2 - 2\partial_- \hat{K} = t^2(\partial_x^2 - 2\partial_- \partial_t) + 2t\left(\frac{1}{2} - z\partial_z\right)\partial_- - z^2\partial_-^2. \quad (2.43)$$

We could use the one-point invariant variable obtained from u by setting in (1.202) $t' = 0, x' = 0, x'_- = 0, z' = 1$, i. e. we use

$$\tilde{u} = \frac{4z}{x^2 - 2tx_- + (z+1)^2}. \quad (2.44)$$

Substituting this change in (2.43) we obtain

$$\hat{S} = \frac{t^2}{z} \left(\tilde{u}^3 \partial_{\tilde{u}} + \frac{\tilde{u}^4}{2} \frac{\partial^2}{\partial \tilde{u}^2} \right). \quad (2.45)$$

We shall elaborate on the use of (2.45) elsewhere.

Now we set an Ansatz for the fields in the bulk: $\phi(t, x, x_-, z) = e^{Mx_-} \phi(t, x, z)$, which leads to the identification $\partial_{x_-} = M$ both in the bulk and on the boundary. Thus, we shall use

$$\hat{S}_0 \doteq \hat{G}_0^2 - 2M\hat{K}_0 = t^2(\partial_x^2 - 2M\partial_t) + 2tM\left(\frac{1}{2} - z\partial_z\right) - z^2M^2. \quad (2.46)$$

Thus, we obtain the following *Schrödinger-like equation* in the bulk:

$$\hat{S}_0\phi = \phi', \quad \phi \in \hat{C}^{1/2}, \quad \phi' \in \hat{C}^{5/2}. \quad (2.47)$$

The relation to the Schrödinger equation on the boundary is seen by the following commutative diagram:

$$\begin{array}{ccc} \hat{C}^{1/2} & \xrightarrow{\hat{S}_0} & \hat{C}^{5/2} \\ \downarrow L_{1/2} & & \downarrow L_{1/2} \\ C^{1/2} & \xrightarrow{S} & C^{5/2} \end{array} \quad (2.48)$$

where $L_{1/2}$ is the bulk-to-boundary operator defined in (1.208), and (2.48) may be re-written as the intertwining relation:

$$S \circ L_{1/2} = L_{1/2} \circ \hat{S}_0, \quad \text{acting as operator } \hat{C}^{1/2} \longrightarrow C^{5/2}. \quad (2.49)$$

Equation (2.49) and therefore (2.48) follow by substitution of the definitions.

As expected, we have a *Schrödinger-like hierarchy of equations* in the bulk:

$$(\hat{S}_0)^{p/2}\phi = \phi', \quad \phi \in \hat{C}^{(3-p)/2}, \quad \phi' \in \hat{C}^{(3+p)/2}, \quad p \in 2\mathbb{N}. \quad (2.50)$$

They are equivalent to the Schrödinger hierarchy of equations on the boundary (2.40) which is proved by showing the analogues of (2.48) and (2.49):

$$\begin{array}{ccc} \hat{C}^\Delta & \xrightarrow{(\hat{S}_0)^{p/2}} & \hat{C}^{3-\Delta} \\ \downarrow L_\Delta & & \downarrow L_\Delta \\ C^\Delta & \xrightarrow{S^{p/2}} & C^{3-\Delta} \end{array} \quad (2.51)$$

$$\begin{aligned} S^{p/2} \circ L_\Delta &= L_\Delta \circ (\hat{S}_0)^{p/2}, \quad \text{acting as operator } \hat{C}^\Delta \longrightarrow C^{3-\Delta}, \\ \Delta &= (3-p)/2, \quad p \in 2\mathbb{N}. \end{aligned} \quad (2.52)$$

2.2 Non-relativistic invariant differential equations for arbitrary n

In this section we review references [8, 190].

2.2.1 Gauss decomposition of the Schrödinger group

Triangular decomposition of $\hat{\mathcal{S}}(n)$ for $n = 2N$

We have

$$\begin{aligned}\hat{\mathcal{S}}(2N) &= \hat{\mathcal{S}}(2N)^+ \oplus \hat{\mathcal{S}}(2N)^0 \oplus \hat{\mathcal{S}}(2N)^-, \\ \hat{\mathcal{S}}(2N)^+ &= \text{l.s.}\{G_a, K, E_{\ell_i \pm \ell_j}\}, \\ \hat{\mathcal{S}}(2N)^0 &= \text{l.s.}\{D, M, H_k\}, \\ \hat{\mathcal{S}}(2N)^- &= \text{l.s.}\{P_a, P_t, E_{-(\ell_i \pm \ell_j)}\},\end{aligned}\tag{2.53}$$

where l. s. stands for linear span, a, k, i, j are integers of $1 \leq a \leq 2N, 1 \leq k \leq N, 1 \leq i < j \leq N$. For $N > 1$ the generators $\{H_k\}, \{E_{\ell_i \pm \ell_j}\}, \{E_{-(\ell_i \pm \ell_j)}\}$ are the Cartan subalgebra generators, the positive root vectors and the negative root vectors of $\text{so}(2N)$, respectively. They are related to the antisymmetric generators $\{J_{ij}\}$ as follows (only the first line remaining for $N = 1$):

$$\begin{aligned}H_k &= -iJ_{k\ N+k} \quad (k = 1, 2, \dots, N), \\ E_{\pm(\ell_j + \ell_k)} &= -\frac{1}{2}(J_{jk} \mp iJ_{j\ N+k} \pm iJ_{k\ N+j} - J_{N+j\ N+k}) \quad (1 \leq j < k \leq N), \\ E_{\ell_j - \ell_k} &= -\frac{1}{2}(J_{jk} + iJ_{j\ N+k} + iJ_{k\ N+j} + J_{N+j\ N+k}) \quad (1 \leq j \neq k \leq N).\end{aligned}\tag{2.54}$$

Triangular decomposition of $\hat{\mathcal{S}}(n)$ for $n = 2N + 1$

We have

$$\begin{aligned}\hat{\mathcal{S}}(2N + 1) &= \hat{\mathcal{S}}(2N + 1)^+ \oplus \hat{\mathcal{S}}(2N + 1)^0 \oplus \hat{\mathcal{S}}(2N + 1)^-, \\ \hat{\mathcal{S}}(2N + 1)^+ &= \text{l.s.}\{G_a, K, E_{\ell_k}, E_{\ell_i \pm \ell_j}\}, \\ \hat{\mathcal{S}}(2N + 1)^0 &= \text{l.s.}\{D, M, H_k\}, \\ \hat{\mathcal{S}}(2N + 1)^- &= \text{l.s.}\{P_a, P_t, E_{-\ell_k}, E_{-(\ell_i \pm \ell_j)}\},\end{aligned}\tag{2.55}$$

where a, k, i, j are integers of $1 \leq a \leq 2N + 1, 1 \leq k \leq N, 1 \leq i < j \leq N$. The generators $\{H_k\}, \{E_{\ell_k}, E_{\ell_i \pm \ell_j}\}, \{E_{-\ell_k}, E_{-(\ell_i \pm \ell_j)}\}$ are the Cartan subalgebra generators, the positive root vectors and the negative root vectors of $\text{so}(2N + 1)$, respectively. They are related to the antisymmetric generators $\{J_{ij}\}$ as in the case $n = 2N$ for $H_k, E_{\pm(\ell_j + \ell_k)}$, and $E_{\ell_j - \ell_k}$, while $E_{\pm\ell_k}$ is defined by

$$E_{\pm\ell_k} = -\frac{1}{\sqrt{2}}(J_{k\ 2N+1} \mp iJ_{N+k\ 2N+1}) \quad (k = 1, 2, \dots, N).\tag{2.56}$$

Gauss decomposition of the Schrödinger group $\hat{\mathbf{S}}(n)$

Let $g \in \hat{\mathbf{S}}(n)$ be an element with Gauss decomposition: $g = g_+ g_0 g_-$.

For even $n = 2N$ we have

$$\begin{aligned} g_+ &= \left(\prod_{a=1}^{2N} e^{x_a G_a} \right) e^{tK} \left(\prod_{1 \leq j < k \leq N} e^{\xi_{jk} E_{\ell_j + \ell_k}} \right) \left(\prod_{1 \leq j < k \leq N} e^{\eta_{jk} E_{\ell_j - \ell_k}} \right), \\ g_0 &= e^{\delta D} e^{mM} \prod_{k=1}^N e^{h_k H_k}, \\ g_- &= \left(\prod_{a=1}^{2N} e^{u_a P_a} \right) e^{\kappa P_t} \left(\prod_{1 \leq j < k \leq N} e^{\xi_{jk}^- E_{-\ell_j - \ell_k}} \right) \left(\prod_{1 \leq j < k \leq N} e^{\eta_{jk}^- E_{-\ell_j + \ell_k}} \right). \end{aligned} \quad (2.57)$$

For odd $n = 2N + 1$ we have

$$\begin{aligned} g_+ &= \left(\prod_{a=1}^{2N+1} e^{x_a G_a} \right) e^{tK} \left(\prod_{k=1}^N e^{\varphi_k E_{\ell_k}} \right) \\ &\quad \times \left(\prod_{1 \leq j < k \leq N} e^{\xi_{jk} E_{\ell_j + \ell_k}} \right) \left(\prod_{1 \leq j < k \leq N} e^{\eta_{jk} E_{\ell_j - \ell_k}} \right), \\ g_0 &= e^{\delta D} e^{mM} \prod_{k=1}^N e^{h_k H_k}, \\ g_- &= \left(\prod_{a=1}^{2N+1} e^{u_a P_a} \right) e^{\kappa P_t} \left(\prod_{k=1}^N e^{\varphi_k^- E_{-\ell_k}} \right) \\ &\quad \times \left(\prod_{1 \leq j < k \leq N} e^{\xi_{jk}^- E_{-\ell_j - \ell_k}} \right) \left(\prod_{1 \leq j < k \leq N} e^{\eta_{jk}^- E_{-\ell_j + \ell_k}} \right). \end{aligned} \quad (2.58)$$

2.2.2 Representations of $\hat{\mathbf{S}}(n)$

Let briefly recall the SSLG setting [161] adapted to the Schrödinger setting above.

- $\mathcal{C}_\Lambda$: the space of $\mathbb{C}^\infty$ functions $\mathcal{F}$ on $\hat{\mathbf{S}}(n)$ with right covariance property

$$\mathcal{F}(xg'g) = e^{\Lambda(H)} \mathcal{F}(g), \quad (2.59)$$

where $x = e^H \in g_0$, $H \in \hat{\mathfrak{s}}^0(n)$, $g' \in g_-$, $\Lambda \in (\hat{\mathfrak{s}}^0(n))^*$. We use the following notation for the values of $\Lambda(H)$: $\Lambda(M) = -m$, $\Lambda(D) = \Delta$, $\Lambda(H_k) = -h_k \in \mathbb{Z}$.

- $\mathcal{C}_\Lambda$: space of restricted functions $\psi = \mathcal{F}|_{g_+}$.
- $\pi_L(X)$: left action of $X \in \hat{\mathfrak{s}}(n)$ on $\mathcal{C}_\Lambda$

$$\pi_L(X) \mathcal{F}(g) = \left. \frac{d}{d\tau} \mathcal{F}(e^{-\tau X} g) \right|_{\tau=0}. \quad (2.60)$$

- $\pi_R(X)$: right action of $X \in \hat{\mathfrak{s}}(n)$ on $\mathcal{C}_\Lambda$

$$\pi_R(X) \mathcal{F}(g) = \left. \frac{d}{d\tau} \mathcal{F}(g e^{\tau X}) \right|_{\tau=0}. \quad (2.61)$$

Representations for $n = 2N$

Define

$$G_k^\pm = G_k \pm iG_{N+k}, \quad x_k^\pm = \frac{1}{2}(x_k \mp ix_{N+k}), \quad k = 1, 2, \dots, N; \quad (2.62)$$

then

$$\prod_{a=1}^{2N} e^{x_a G_a} = \prod_{k=1}^N e^{x_k^+ G_k^+} e^{x_k^- G_k^-}. \quad (2.63)$$

Sometimes the use of (2.63) is more convenient. We denote each factor of g_+ by

$$g_+ = \Gamma_x e^{tK} \Gamma_\xi \Gamma_\eta, \quad (2.64)$$

$$\Gamma_x = \prod_{a=1}^{2N} e^{x_a G_a}, \quad \Gamma_\xi = \prod_{1 \leq j < k \leq N} e^{\xi_{jk} E_{\ell_j + \ell_k}}, \quad \Gamma_\eta = \prod_{1 \leq j < k \leq N} e^{\eta_{jk} E_{\ell_j - \ell_k}}.$$

Action of $\hat{S}(2N)^0$

Following the general procedure of [161] and results of [8, 190] we obtain the following formulae for the vector-field representation:

$$\pi_L(M) = -\Lambda(M), \quad \pi_L(D) = -\Lambda(D) - \sum_{a=1}^n x_a \partial_{x_a} - 2t \partial_t, \quad (2.65)$$

$$\begin{aligned} \pi_L(H_k) = & -\Lambda(H_k) + x_k^+ \partial_{x_k^+} - x_k^- \partial_{x_k^-} \\ & - \sum_{i=1}^{k-1} (\xi_{ik} \partial_{\xi_{ik}} - \eta_{ik} \partial_{\eta_{ik}}) - \sum_{j=k+1}^N (\xi_{kj} \partial_{\xi_{kj}} + \eta_{kj} \partial_{\eta_{kj}}). \end{aligned} \quad (2.66)$$

The x -dependent parts of (2.66) can be rewritten as

$$i(x_k \partial_{x_{N+k}} - x_{N+k} \partial_{x_k}). \quad (2.67)$$

Action of $\hat{S}(2N)^+$ [8]

$$\pi_L(G_a) = -\partial_{x_a}, \quad \pi_L(K) = -\partial_t, \quad (2.68)$$

$$\pi_L(E_{\ell_j + \ell_k}) = x_j^+ \partial_{x_k^-} - x_k^+ \partial_{x_j^-} - \partial_{\xi_{jk}}, \quad (2.69)$$

$$\begin{aligned} \pi_L(E_{\ell_j - \ell_k}) = & x_j^+ \partial_{x_k^+} - x_k^- \partial_{x_j^-} - \sum_{s=k+1}^N \xi_{ks} \partial_{\xi_{js}} - \sum_{r=1}^{j-1} \xi_{rk} \partial_{\xi_{rj}} \\ & + \sum_{r=j+1}^{k-1} \xi_{rk} \partial_{\xi_{jr}} + \sum_{r=1}^{j-1} \eta_{rj} \partial_{\eta_{rk}} - \partial_{\eta_{jk}}. \end{aligned} \quad (2.70)$$

Action of $\hat{\mathcal{S}}(2N)^-$

Standardly, the left action is determined by the formula

$$e^{-\tau X} g_+ g_0 g_- = g'_+ g'_0 g'_-, \quad (2.71)$$

where $X \in \mathfrak{g}_-$. For us, it will be enough to find g'_+ and g'_0 because the vector-field representation is restricted to $\mathfrak{g}_+$.

We introduce the generators $P_k^\pm \doteq P_k \pm iP_{N+k}$. We have

$$\begin{aligned} e^{-\tau P_k^+} g &= \left(\prod_{r=1}^N e^{x_r'^+ G_r^+} e^{x_r'^- G_r^-} \right) e^{tK} \left(\prod_{r<s} e^{\xi_{rs} E_{\ell_r+\ell_s}} \right) \left(\prod_{r<s} e^{\eta_{rs} E_{\ell_r-\ell_s}} \right) \\ &\quad \times e^{\delta D} e^{m'M} \left(\prod_{r=1}^N e^{h_r H_r} \right) g'_-, \end{aligned} \quad (2.72)$$

$$x_k'^+ = x_k^+ - \tau t, \quad x_r'^\pm = x_r^\pm \text{ (otherwise)}, \quad m' = m - 2\tau x_k^-,$$

$$\begin{aligned} e^{-\tau P_k^-} g &= \left(\prod_{r=1}^N e^{x_r'^+ G_r^+} e^{x_r'^- G_r^-} \right) e^{tK} \left(\prod_{r<s} e^{\xi_{rs} E_{\ell_r+\ell_s}} \right) \left(\prod_{r<s} e^{\eta_{rs} E_{\ell_r-\ell_s}} \right) \\ &\quad \times e^{\delta D} e^{m'M} \left(\prod_{r=1}^N e^{h_r H_r} \right) g'_-, \end{aligned} \quad (2.73)$$

$$x_k'^- = x_k^- - \tau t, \quad x_r'^\pm = x_r^\pm \text{ (otherwise)}, \quad m' = m - 2\tau x_k^+,$$

$$\begin{aligned} e^{-\tau P_i} g &= \left(\prod_{r=1}^N e^{x_r'^+ G_r^+} e^{x_r'^- G_r^-} \right) e^{t'K} \left(\prod_{r<s} e^{\xi_{rs} E_{\ell_r+\ell_s}} \right) \left(\prod_{r<s} e^{\eta_{rs} E_{\ell_r-\ell_s}} \right) \\ &\quad \times e^{\delta' D} e^{m'M} \left(\prod_{r=1}^N e^{h_r H_r} \right) g'_-, \end{aligned} \quad (2.74)$$

$$x_j'^+ = x_j^+ - \tau t x_j^+, \quad x_j'^- = x_j^- - \tau t x_j^-, \quad t' = t - \tau t^2,$$

$$\delta' = \delta - \tau t, \quad m' = m - 2\tau \sum_{j=1}^N x_j^+ x_j^-,$$

$$\begin{aligned} e^{-\tau E_{-(\ell_j+\ell_k)}} g &= \left(\prod_{r=1}^N e^{x_r'^+ G_r^+} e^{x_r'^- G_r^-} \right) e^{tK} \left(\prod_{r<s} e^{\xi'_{rs} E_{\ell_r+\ell_s}} \right) \left(\prod_{r<s} e^{\eta'_{rs} E_{\ell_r-\ell_s}} \right) \\ &\quad \times e^{\delta D} e^{mM} \left(\prod_{r=1}^N e^{h'_r H_r} \right) g'_-, \end{aligned} \quad (2.75)$$

$$x_i'^+ = x_i^+ - \tau(\delta_{ij} x_k^- - \delta_{ik} x_j^-), \quad x_i'^- = x_i^-,$$

$$\xi'_{rs} = \xi_{rs} - \delta_{jr} \delta_{ks} \xi_{rs}^2, \quad \xi'_{sr} = \xi_{sr} - \delta_{js} \delta_{kr} \xi_{sr}^2$$

$$\eta'_{rs} = \eta_{rs} - \tau\{\delta_{sk}(\xi_{ir} - \xi_{ri}) + \delta_{sj}(\xi_{rk} - \xi_{kr})\},$$

$$h'_l = h_l - \tau(\delta_{jl} + \delta_{kl}) \xi_{jk},$$

$$\begin{aligned}
 e^{-\tau E_{-(\ell_j - \ell_k)}} g &= \left(\prod_{r=1}^N e^{x_r'^+ G_r^+} e^{x_r'^- G_r^-} \right) e^{tK} \left(\prod_{r < s} e^{\xi_{rs}' E_{\ell_r + \ell_s}} \right) \left(\prod_{r < s} e^{\eta_{rs}' E_{\ell_r - \ell_s}} \right) \\
 &\quad \times e^{\delta D} e^{mM} \left(\prod_{r=1}^N e^{h_r' H_r} \right) g_{-}', \quad (2.76) \\
 x_i'^+ &= x_i^+ + \tau \delta_{ij} x_k^+, \quad x_i'^- = x_i^- - \tau \delta_{ik} x_j^-, \\
 \xi_{rs}' &= \xi_{rs} - \tau \delta_{kr} \xi_{is} - \tau \delta_{sk} \xi_{ri}, \quad h_l' = h_l - \tau (\delta_{kl} - \delta_{il}) \xi_{jk}, \\
 \eta_{rs}' &= \eta_{rs} - \tau \delta_{rk} \eta_{js} + \tau \delta_{js} \eta_{rk} - \delta_{rj} \delta_{sk} \eta_{rs}^2.
 \end{aligned}$$

The above formulae are correct up to terms quadratic in τ . This is enough since these formulae are used only to obtain the left action of the corresponding generators of the algebra (see formulae (2.60)).

Thus, from the infinitesimal left action we obtain the vector-field representation:

$$\begin{aligned}
 \pi_L(P_k^+) &= -t \partial_{x_k^+} - 2\Lambda(M) x_k^-, \quad (2.77) \\
 \pi_L(P_k^-) &= -t \partial_{x_k^-} - 2\Lambda(M) x_k^+, \\
 \pi_L(P_t) &= -t^2 \partial_t - t \sum_{j=1}^N (x_j^+ \partial_{x_j^+} + x_j^- \partial_{x_j^-}) - 2\Lambda(M) \sum_{j=1}^N x_j^+ x_j^- - \Lambda(D)t, \\
 \pi_L(E_{-(\ell_j + \ell_k)}) &= -(\Lambda(H_j) + \Lambda(H_k)) \xi_{jk} + x_j^+ \partial_{x_k^-} - x_k^+ \partial_{x_j^-} - \xi_{ik}^2 \partial_{\xi_{ik}} \\
 &\quad - \sum_{p=j+1}^{k-1} \xi_{jp} \partial_{\eta_{pk}} - \sum_{p=1}^{j-1} \xi_{pk} \partial_{\eta_{pj}} + \sum_{p=1}^{j-1} \xi_{pj} \partial_{\eta_{pk}}, \\
 \pi_L(E_{-(\ell_j - \ell_k)}) &= -(\Lambda(H_k) - \Lambda(H_j)) \xi_{jk} + x_k^+ \partial_{x_j^+} - x_j^- \partial_{x_k^-} + \eta_{jk}^2 \partial_{\eta_{jk}} \\
 &\quad - \sum_{s=j+1}^{k-1} \xi_{js} \partial_{\xi_{sk}} + \sum_{s=k+1}^N \xi_{js} \partial_{\xi_{ks}} \\
 &\quad - \sum_{r=1}^{j-1} \xi_{rj} \partial_{\xi_{rk}} - \sum_{q=k+1}^N \eta_{jq} \partial_{\eta_{kq}}.
 \end{aligned}$$

Representations for $n = 2N + 1$

In addition to $G_k^\pm, x_k^\pm$ given by (2.62), let us introduce

$$G_0 = \sqrt{2} G_{2N+1}, \quad x_0 = \frac{1}{\sqrt{2}} x_{2N+1}, \quad (2.78)$$

then

$$\prod_{a=1}^{2N+1} e^{x_a G_a} = e^{x_0 G_0} \prod_{k=1}^N e^{x_k^+ G_k^+} e^{x_k^- G_k^-}. \quad (2.79)$$

The difference from $n = 2N$ is the existence of G_0 and E_{ℓ_k} .

Action of $\hat{\mathcal{S}}(2N + 1)^0$

Since $[D, G_0] = D_0$, $[D, E_{\ell_k}] = 0$, the left representation of M , D are the same as in the n -even case; cf. (2.65), while for H_k we have

$$\pi_L(H_k) = \pi_L(H_k)^{\text{even}} - \varphi_k \partial_{\varphi_k} \quad (2.80)$$

where $\pi_L(H_k)^{\text{even}}$ is the RHS of formula (2.66).

Action of $\hat{\mathcal{S}}(2N + 1)^+$

The representations of G_a , K , $E_{\ell_j + \ell_k}$ are the same as in the n -even case, cf. (2.68) with $a = 1, 2, \dots, n = 2N + 1$, while for the others we have [8]

$$\pi_L(E_{\ell_k}) = x_k^+ \partial_{x_0} - x_0 \partial_{x_k^-} - \partial_{\varphi_k} - \sum_{i=1}^{k-1} \varphi_i \partial_{\xi_{ik}} + \sum_{j=k+1}^N \varphi_j \partial_{\xi_{kj}}, \quad (2.81)$$

$$\pi_L(E_{\ell_j - \ell_k}) = \pi_L(E_{\ell_j - \ell_k})^{\text{even}} - \varphi_k \partial_{\varphi_j}, \quad (2.82)$$

where $\pi_L(E_{\ell_j - \ell_k})^{\text{even}}$ is the RHS of (2.70).

Action of $\hat{\mathcal{S}}(2N + 1)^-$

Since $P_k^\pm$ commute with G_0 and $[P_k^+, E_{\ell_k}] = P_0$ gives rise only to changing of g_- , the left action of $P_k^\pm$ is the same as in the even case. Next,

$$[P_t, G_0^n] = nG_0^{n-1} + n(n-1)G_0^{n-2}M, \quad [P_t, E_{\ell_k}] = -P_k^- \quad (2.83)$$

only leads to

$$m' = m'_{\text{even}} - \tau \chi_0^2,$$

from which it follows that

$$\pi_L(P_t) = \pi_L(P_t)^{\text{even}} - \Lambda(M) \chi_0^2. \quad (2.84)$$

For the left action of $E_{-(\ell_j - \ell_k)}$ the unique non-vanishing commutator is

$$[E_{-(\ell_j - \ell_k)}, E_{\ell_k}] = E_{\ell_k},$$

which means that

$$\varphi'_z = \varphi_z - \tau \delta_{zk} \varphi_j$$

and consequently

$$\pi_L(E_{-(\ell_j - \ell_k)}) = \pi_L(E_{-(\ell_j - \ell_k)})^{\text{even}} - \varphi_j \partial_{\varphi_k}. \quad (2.85)$$

For the rest of the generators we first obtain

$$\begin{aligned}
 e^{-\tau P_0} g &= e^{x'_0 G_0} \left(\prod_{r=1}^N e^{x'^+_r G_r^+} e^{x'^-_r G_r^-} \right) e^{tK} \left(\prod_{r<s} e^{\xi'_{rs} E_{\ell_r + \ell_s}} \right) \left(\prod_{r<s} e^{\eta'_{rs} E_{\ell_r - \ell_s}} \right) \\
 &\quad \times e^{\delta D} e^{m' M} \prod_{r=1}^N e^{h'_r H_r} g'_-, \\
 x'_0 &= x_0 - \tau t, \quad m' = m - 2\tau x_0,
 \end{aligned} \tag{2.86}$$

$$\begin{aligned}
 e^{-\tau E_{-\ell_k}} g &= e^{x'_0 G_0} \left(\prod_{r=1}^N e^{x'^+_r G_r^+} e^{x'^-_r G_r^-} \right) e^{tK} \left(\prod_{r=1}^N e^{\varphi'_r E_{\ell_r}} \right) \\
 &\quad \times \left(\prod_{r<s} e^{\xi'_{rs} E_{\ell_r + \ell_s}} \right) \left(\prod_{r<s} e^{\eta'_{rs} E_{\ell_r - \ell_s}} \right) e^{\delta D} e^{m' M} \left(\prod_{r=1}^N e^{h'_r H_r} \right) g'_-, \\
 x'_0 &= x_0 + \tau x_k^-, \quad x'^+_r = x_r^+ - \tau \delta_{rk} x_0, \\
 \varphi'_z &= \varphi_z - \tau \delta_{zk} \frac{\varphi_k^2}{2} + \tau (\delta_{zs} \xi_{ks} + \delta_{zr} \xi_{rk} - \delta_{qz} \eta_{kq}), \\
 \eta'_{pq} &= \eta_{pq} - \tau \delta_{zp} \delta_{kq} \varphi_z, \quad h'_z = h_z - \tau \delta_{kz} \varphi_z,
 \end{aligned} \tag{2.87}$$

$$\begin{aligned}
 e^{-\tau E_{-(\ell_j + \ell_k)}} g &= e^{x'_0 G_0} \left(\prod_{r=1}^N e^{x'^+_r G_r^+} e^{x'^-_r G_r^-} \right) e^{tK} \left(\prod_{r=1}^N e^{\varphi'_r E_{\ell_r}} \right) \\
 &\quad \times \left(\prod_{r<s} e^{\xi'_{rs} E_{\ell_r + \ell_s}} \right) \left(\prod_{r<s} e^{\eta'_{rs} E_{\ell_r - \ell_s}} \right) e^{\delta D} e^{m' M} \left(\prod_{r=1}^N e^{h'_r H_r} \right) g'_-, \\
 x'^+_r &= (x'^+_r)^{\text{even}}, \quad x'^-_r = (x'^-_r)^{\text{even}}, \\
 \xi'_{rs} &= (\xi'_{rs})^{\text{even}}, \quad \eta'_{rs} = (\eta'_{rs})^{\text{even}}, \quad h'_l = (h'_l)^{\text{even}}, \\
 \varphi'_z &= \varphi_z - \tau (\delta_{jz} \varphi_z \xi_{zk} - \delta_{kz} \varphi_z \xi_{jz}).
 \end{aligned} \tag{2.88}$$

From the above the vector-field representation follows:

$$\pi_L(P_0) = -t \partial_{x_0} - 2x_0 \Lambda(M), \tag{2.89}$$

$$\begin{aligned}
 \pi_L(E_{-\ell_k}) &= -\varphi_k \Lambda(H_k) + x_k^- \partial_{x_0} - x_0 \partial_{x_k^-} \\
 &\quad - \frac{\varphi_k^2}{2} \partial_{\varphi_k} + \sum_{s=k+1}^N \xi_{ks} \partial_{\varphi_s} + \sum_{r=1}^{k-1} \xi_{rk} \partial_{\varphi_r} \\
 &\quad - \sum_{q=k+1}^N \eta_{kq} \partial_{\varphi_q} - \sum_{p=1}^{k-1} \varphi_p \partial_{\eta_{pk}},
 \end{aligned} \tag{2.90}$$

$$\pi_L(E_{-(\ell_j + \ell_k)}) = \pi_L(E_{-(\ell_j + \ell_k)})^{\text{even}} - \xi_{jk} (\varphi_j \partial_{\varphi_j} - \varphi_k \partial_{\varphi_k}). \tag{2.91}$$

2.2.3 Singular vectors and invariant equations for $\hat{\mathcal{S}}(2N)$

Singular vectors

Let v_0 be the lowest weight vector of the Verma module V^Λ with lowest weight Λ :

$$Hv_0 = \Lambda(H)v_0, \quad Xv_0 = 0, \quad \text{for } \forall H \in \mathfrak{g}^0, \forall X \in \mathfrak{g}^-. \quad (2.92)$$

A singular vector is a vector v_s in the Verma module V^Λ such that

$$Hv_s = \Lambda'(H)v_s, \quad Xv_s = 0, \quad \text{for } \forall H \in \mathfrak{g}^0, \forall X \in \mathfrak{g}^-, \Lambda' \neq \Lambda. \quad (2.93)$$

Since the singular vectors are in $V^\Lambda \simeq U(\mathfrak{g})v_0$, let us introduce [8]

$$\begin{aligned} v_{\alpha\beta\gamma\lambda\rho} = & \left\{ \prod_{r=1}^N (G_r^+)^{\alpha_r} \right\} \left\{ \prod_{r=1}^N (G_r^-)^{\beta_r} \right\} K^\gamma \left\{ \prod_{1 \leq r \leq N} (E_{\ell_r + \ell_s})^{\lambda_{rs}} \right\} \\ & \times \left\{ \prod_{1 \leq r \leq N} (E_{\ell_r - \ell_s})^{\rho_{rs}} \right\} v_0, \end{aligned} \quad (2.94)$$

where α, β, λ and ρ are abbreviations of the sets of nonnegative integers

$$\begin{aligned} \alpha &= [\alpha_1, \alpha_2, \dots, \alpha_N], \quad \beta = [\beta_1, \beta_2, \dots, \beta_N], \\ \lambda &= [\lambda_{12}, \lambda_{13}, \dots, \lambda_{N-1N}], \quad \rho = [\rho_{12}, \rho_{13}, \dots, \rho_{N-1N}], \end{aligned} \quad (2.95)$$

and γ is also a nonnegative integer.

Elementary computation shows that the action of $\mathfrak{g}^0$ on $v_{\alpha\beta\gamma\lambda\rho}$ is given by

$$\begin{aligned} Mv_{\alpha\beta\gamma\lambda\rho} &= \Lambda(M)v_{\alpha\beta\gamma\lambda\rho} = -mv_{\alpha\beta\gamma\lambda\rho}, \\ Dv_{\alpha\beta\gamma\lambda\rho} &= (\Lambda(D) + p)v_{\alpha\beta\gamma\lambda\rho} = (p - d)v_{\alpha\beta\gamma\lambda\rho}, \\ H_kv_{\alpha\beta\gamma\lambda\rho} &= (\Lambda(H_k) + r_k)v_{\alpha\beta\gamma\lambda\rho} = (r_k - h_k)v_{\alpha\beta\gamma\lambda\rho}, \end{aligned} \quad (2.96)$$

$$p = \sum_{r=1}^N (\alpha_r + \beta_r) + 2\gamma, \quad (2.97)$$

$$r_k = -\alpha_k + \beta_k + \sum_{s=k+1}^N (\lambda_{ks} + \rho_{ks}) + \sum_{r=1}^{k-1} (\lambda_{rk} - \rho_{rk}).$$

The singular vectors have the following form:

$$v_s = \sum_{\alpha, \lambda, \rho} a_{\alpha\lambda\rho} v_{\alpha\beta\gamma\lambda\rho} \quad (2.98)$$

where we fix the values of p, r_k ($k = 1, 2, \dots, N$) to make $v_{\alpha\beta\gamma\lambda\rho}$ homogeneous. Let us look for the singular vectors with nonzero central element: $\Lambda(M) \neq 0$.

Applying $Xv_s = 0$ (2.93) for $X = P_k^\pm$, $X = P_t$, $X = E_{-(\ell_i \pm \ell_j)} v_s$ ($i < j$), in [190] the general explicit formula for the singular vectors was obtained:

$$v_s = c \left(\sum_{a=1}^{2N} G_a^2 - 2\Lambda(M)K \right)^{p/2} \sum_{\lambda, \varrho} \left(\prod_{r < s} E_{\ell_r + \ell_s}^{\lambda_{rs}} \prod_{p < q} E_{\ell_p - \ell_q}^{\varrho_{pq}} \right) v_0. \quad (2.99)$$

The example $N = 1$ is trivial, since (2.99) reduces just to

$$v_s = c \left(\sum_{a=1}^2 G_a^2 - 2\Lambda(M)K \right)^{p/2} v_0. \quad (2.100)$$

Example: $N = 2$

We have $i = 1, j = 2, r_1 = \lambda_{12} + \rho_{12}, r_2 = \lambda_{12} - \rho_{12}$ are fixed, and

$$\lambda_{12} = (1 + h_1 + h_2)/3, \quad \rho_{12} = (h_1 - h_2 - 1)/3.$$

Consequently the form of the singular vectors is

$$v_s = c \left(\sum_{a=1}^4 G_a^2 - 2\Lambda(M)K \right)^{p/2} (1 + E_{\ell_1 + \ell_2}^{\lambda_{12}} + E_{\ell_1 - \ell_2}^{\rho_{12}} + E_{\ell_1 + \ell_2}^{\lambda_{12}} E_{\ell_1 - \ell_2}^{\rho_{12}}) v_0. \quad (2.101)$$

Example: $N = 3$

1: $i = 1, j = 2$

From the extra conditions listed above, it follows that $\lambda_{13} = 0, \rho_{13} = 0$.

2: $i = 1, j = 2$

In this case we have $\lambda_{12} = 0$.

3: $i = 2, j = 3$

From the extra conditions listed above, it follows that $\lambda_{12} = 0, \rho_{13} = 0$.

This leads to

$$r_1 = \rho_{12}, \quad r_2 = \lambda_{23} + \rho_{23} - \rho_{12}, \quad r_3 = \lambda_{23} - \rho_{23}. \quad (2.102)$$

On the other hand

$$\rho_{12} = 1 + h_1 - h_2 + r_2 - r_1, \quad \lambda_{23} = 1 + h_2 + h_3 - r_2 - r_3, \quad \rho_{23} = 1 + h_2 - h_3 + r_3 - r_2. \quad (2.103)$$

It follows that

$$\rho_{12} = (3h_1 - h_2 + 5)/7, \quad \rho_{23} = (h_1 + 2h_2 + 4)/7 - h_3/3, \quad \lambda_{23} = (h_1 + 2h_2 + 4)/7 + h_3/3.$$

The most general form of the singular vector for $n = 2N = 6$ is

$$v_s = c \left(\sum_{a=1}^6 G_a^2 - 2\Lambda(M)K \right)^{p/2} \sum_{q=0, \lambda_{23}; r=0, \rho_{12}; s=0, \rho_{23}} E_{\ell_2 + \ell_3}^q E_{\ell_1 - \ell_2}^r E_{\ell_2 - \ell_3}^s v_0. \quad (2.104)$$

Invariant equations

Following the general procedure of [161] to obtain the invariant operators and equations we must substitute any generator X in the expression (2.99) for the singular vector with the right action $\pi_R(X)$ given in [8]. Thus, we obtain the invariant equations

$$\left(\sum_{a=1}^{2N} \frac{\partial^2}{\partial x_a^2} + 2M\partial_t \right)^{p/2} \sum_{\lambda, \rho} \left(\prod_{r < s} \pi_R(E_{\ell_r + \ell_s})^{\lambda_{rs}} \prod_{p < q} \pi_R(E_{\ell_p - \ell_q})^{\rho_{pq}} \right) \psi = \psi' \quad (2.105)$$

where $\pi_R(E_{\ell_r + \ell_s})$, $\pi_R(E_{\ell_p - \ell_q})$ are given by formulae (4.27) and (4.28) of [8], $m = -\Lambda(M)$, and ψ' belongs to the representation Λ' such that $\Lambda'(M) = \Lambda(M)$, $\Lambda'(H_k) = \Lambda(H_k) + r_k$ and $\Lambda'(D) = \Lambda(D) + p = \Delta + p = \frac{1}{2}p + N + 1$.

In particular for $N = 2$ one obtains

$$\left(\sum_{a=1}^4 \frac{\partial^2}{\partial x_a^2} + 2M\partial_t \right)^{p/2} \sum_{\lambda=0, \lambda_{12}; \rho=0, \rho_{12}} (\partial_{\xi_{12}})^\lambda (\partial_{\eta_{12}})^\rho \psi = \psi', \quad (2.106)$$

while for $N = 3$ the invariant equation is written in the form

$$\left(\sum_{a=1}^6 \frac{\partial^2}{\partial x_a^2} + 2M\partial_t \right)^{p/2} \sum_{q, r, s} (\partial_{\xi_{23}} + \eta_{13}\partial_{\xi_{12}})^q (\partial_{\eta_{12}} + \eta_{23}\partial_{\eta_{13}})^r (\partial_{\eta_{23}})^s \psi = \psi' \quad (2.107)$$

where $q = 0, \lambda_{23}$; $r = 0, \rho_{12}$; $s = 0, \rho_{23}$.

2.2.4 Singular vectors and invariant equations for $\hat{\mathcal{S}}(2N + 1)$

Singular vectors

To be compared with the $n = 2N$ case we have additional elements G_0 and E_{ℓ_k} . Let us introduce

$$\begin{aligned} v_{\omega\alpha\beta\gamma\sigma\lambda\rho} &= G_0^\omega \left\{ \prod_{r=1}^N (G_r^+)^{\alpha_r} \right\} \left\{ \prod_{r=1}^N (G_r^-)^{\beta_r} \right\} K^\gamma \\ &\times \left\{ \prod_{r=1}^N E_{\ell_r}^{\sigma_r} \right\} \left\{ \prod_{1 \leq r < s \leq N} (E_{\ell_r + \ell_s})^{\lambda_{rs}} \right\} \left\{ \prod_{1 \leq r < s \leq N} (E_{\ell_r - \ell_s})^{\rho_{rs}} \right\} v_0 \end{aligned} \quad (2.108)$$

where $\sigma = [\sigma_1, \sigma_2, \dots, \sigma_N]$ is a set of nonnegative integers. The action of M , D , H_k is given as in the $n = 2N$ case, i. e. by (2.96), but the values of p and r_k now are given by

$$p = \omega + \sum_{r=1}^N (\alpha_r + \beta_r) + 2\gamma, \quad (2.109)$$

$$r_k = -\alpha_k + \beta_k + \sigma_k + \sum_{r=k+1}^N (\lambda_{kr} + \rho_{kr}) + \sum_{r=1}^{k-1} (\lambda_{rk} - \rho_{rk}).$$

The singular vectors have the form of

$$v_s = \sum_{\omega, \alpha, \sigma, \lambda, \rho} a_{\omega \alpha \sigma \lambda \rho} v_{\omega \alpha \sigma \lambda \rho}. \quad (2.110)$$

Imposing $Xv_s = 0$ for $X = P_0 \doteq \sqrt{2}P_{2N+1}$, $X = P_k^\pm$, P_0 , $X = E_{-\ell_k}$, $X = E_{-(\ell_i \pm \ell_j)}$ in [190] the general explicit formula for the singular vectors was obtained:

$$v_s = c \left(\sum_{a=1}^{2N+1} G_a^2 - 2\Lambda(M)K \right)^{p/2} \sum_{\sigma=0, \ell_N; \lambda; \rho} \left(E_{\ell_N}^\sigma \prod_{r < s} E_{\ell_r + \ell_s}^{\lambda_{rs}} \prod_{p < q} E_{\ell_p - \ell_q}^{\rho_{pq}} \right) v_0. \quad (2.111)$$

Example: $N = 1$

We have $r_1 = \sigma_1 = 2h_1 - 2r_1$. It follows that $\sigma_1 = r_1 = (2h_1 + 1)/3$ and consequently, cf. [8],

$$v_s = c \left(\sum_{a=1}^3 G_a^2 - 2\Lambda(M)K \right)^{p/2} \sum_{\sigma=0, \sigma_1} E_{\ell_1}^\sigma v_0. \quad (2.112)$$

Example: $N = 2$

We have $i = 1, j = 2$ and $\sigma_1 = 0, \sigma_2 = 2h_2 - 2r_2 + 1$.

Under the conditions which must be satisfied, it follows that $\lambda_{12} = 0$ and $\rho_{12} = r_2 - r_1 + h_1 - h_2 + 1$. On the other hand $r_1 = \rho_{12}, r_2 = \sigma_2 - \rho_{12}$. This leads to

$$\rho_{12} = (6h_1 - h_2 + 3)/5, \quad \sigma_2 = 2(2h_1 + h_2 + 1)/5.$$

Thus, the general form of the singular vector for $n = 2N + 1 = 5$ is

$$v_s = c \left(\sum_{a=1}^5 G_a^2 - 2\Lambda(M)K \right)^{p/2} \sum_{\sigma=0, \sigma_2; \rho=0, \rho_{12}} E_{\ell_2}^\sigma E_{\ell_1 - \ell_2}^\rho v_0. \quad (2.113)$$

Invariant equations

Substituting the right actions into the singular vectors (2.111) and performing a similar computation to $n = 2N$ case, we obtain the invariant equations

$$\begin{aligned} & \left(\sum_{a=1}^{2N+1} \frac{\partial^2}{\partial x_a^2} + 2M\partial_t \right)^{p/2} \sum_{\sigma, \lambda, \rho} (\pi_R(E_{\ell_N}))^\sigma \\ & \times \left(\prod_{r < s} \pi_R(E_{\ell_r + \ell_s})^{\lambda_{rs}} \prod_{p < q} \pi_R(E_{\ell_p - \ell_q})^{\rho_{pq}} \right) \psi = \psi' \end{aligned} \quad (2.114)$$

where $\pi_r(E_{\ell_N}), \pi_R(E_{\ell_r + \ell_s}), \pi_R(E_{\ell_p - \ell_q})$ are given in [8], $m = -\Lambda(M)$, and ψ' belongs to the representation Λ' such that $\Lambda'(M) = \Lambda(M)$, $\Lambda'(H_k) = \Lambda(H_k) + r_k$ and $\Lambda'(D) = \Lambda(D) + p = \Delta + p = \frac{1}{2}(p + 2N + 3)$.

In particular for $N = 1$ one obtains [8], cf. (2.112),

$$\left(\sum_{a=1}^3 \frac{\partial^2}{\partial x_a^2} + 2M\partial_t \right)^{p/2} \sum_{\sigma=0, \sigma_1} (\partial_{\varphi_1})^\sigma \psi = \psi', \quad (2.115)$$

while for $N = 2$ the invariant equation is obtained from (2.113):

$$\left(\sum_{a=1}^5 \frac{\partial^2}{\partial x_a^2} + 2M\partial_t \right)^{p/2} \sum_{\sigma, \rho} (\partial_{\varphi_2} + \eta_{12}(\partial_{\varphi_1} + \partial_{\xi_{12}}))^\sigma (\partial_{\eta_{12}} + \eta_{23}\partial_{\eta_{13}})^\rho \psi = \psi' \quad (2.116)$$

where $\sigma = 0, \sigma_2$; $\rho = 0, \rho_{12}$.

2.3 Non-relativistic invariant equations for $\hat{S}(3)$

2.3.1 Algebraic structure and actions

We consider separately the case $n = 3$ (which is the $N = 1$ odd case of the previous section) since it is most relevant for the physical applications.

Triangular decomposition

$\hat{s}(3) = \hat{s}(3)^+ \oplus \hat{s}(3)^0 \oplus \hat{s}(3)^-$. We have

$$\begin{aligned} \hat{s}(3)^+ &= \{G_0, G_+, G_-, K, E_+\}, \\ \hat{s}(3)^0 &= \{D, M, H\}, \\ \hat{s}(3)^- &= \{P_0, P_+, P_-, P_t, E_-\}. \end{aligned} \quad (2.117)$$

Here we have simplified the notation as follows:

$$\begin{aligned} E_+ &= E_{l_1} = -\frac{1}{\sqrt{2}}(J_{13} - iJ_{23}), \\ E_- &= E_{-l_1} = -\frac{1}{\sqrt{2}}(J_{13} + iJ_{23}), \quad H_1 = H = -J_{12}, \\ G_0 &= \sqrt{2}G_3, \quad G_+ = G_1 + iG_2, \quad G_- = G_1 - iG_2, \\ P_0 &= \sqrt{2}P_3, \quad P_+ = P_1 + iP_2, \quad P_- = P_1 - iP_2. \end{aligned} \quad (2.118)$$

We recall that this complexification reflects on the coordinates which read

$$x_0 = \frac{1}{\sqrt{2}}x_3, \quad x_+ = x_1 - ix_2, \quad x_- = x_1 + ix_2. \quad (2.119)$$

Taking into account (1.174), (2.118) the non-vanishing commutation relations are

$$\begin{aligned} [P_t, D] &= 2P_t, \quad [P_{0,\pm}, D] = P_{0,\pm}, \quad [P_t, G_{0,\pm}] = P_{0,\pm}, \\ [P_t, K] &= D, \quad [P_{0,\pm}, K] = G_{0,\pm}, \quad [P_0, G_0] = [P_\pm, G_\mp] = 2M, \end{aligned}$$

$$\begin{aligned}
[D, G_{0,\pm}] &= G_{0,\pm}, & [D, K] &= 2K, \\
[P_0, E_{\pm}] &= -P_{\mp}, & [P_{\pm}, E_{\pm}] &= P_0, & [P_{\pm}, H] &= \pm P_{\pm}, \\
[G_0, E_{\pm}] &= -G_{\mp}, & [G_{\pm}, E_{\pm}] &= G_0, & [G_{\pm}, H] &= \pm G_{\pm}, \\
[E_{\pm}, H] &= \mp E_{\pm}, & [E_+, E_-] &= -H.
\end{aligned} \tag{2.120}$$

Gauss decomposition

Let $g \in \hat{S}$ be the element of the Schrödinger group, then its triangular decomposition is $g = g_+ g_0 g_-$, where

$$\begin{aligned}
g_+ &= e^{x_0 G_0} e^{x_+ G_+} e^{x_- G_-} e^{tK} e^{\varphi_+ E_+}, \\
g_0 &= e^{\delta D} e^{mM} e^{hH}, \\
g_- &= e^{v_0 P_0} e^{v_+ P_+} e^{v_- P_-} e^{kP_t} e^{\varphi_- E_-}.
\end{aligned} \tag{2.121}$$

Remark 3. The explicit expressions for g_+ , g_0 , g_- can easily be obtained from the following relations valid in the matrix representation (1.181) (in terms of 8×8 matrices):

$$M^q = 0, \quad q > 1, \tag{2.122}$$

$$D^{2q+1} = D, \quad H^{2q+1} = H, \quad q \geq 0, \tag{2.122}$$

$$D^{2q} = D^2, \quad H^{2q} = H^2, \quad q > 0, \tag{2.123}$$

$$G_0^q = G_+^q = G_-^q = G_0 G_+ = G_0 G_- = G_+ G_- = 0, \quad q > 1, \tag{2.124}$$

$$P_0^q = P_+^q = P_-^q = 0, \quad q > 1,$$

$$P_t^q = 0, \quad q > 1,$$

$$K^q = 0, \quad q > 1,$$

$$E_+^q = 0, \quad q > 2,$$

$$E_-^q = 0, \quad q > 2, \tag{2.125}$$

where in the above expressions q is an integer number. Furthermore the matrix realization of the algebra $\hat{s}(3)$ may also be used to obtain the left(right) action.

Left action

- Action of $\hat{s}(3)^0$. Using the commutation relations and matrix realization of $\hat{s}(3)$, up to terms quadratic in τ it follows that

$$\begin{aligned}
e^{-\tau M} g_+ &= g_+ e^{-\tau M}, \\
e^{-\tau D} g_+ &= \prod_{a=1}^3 \exp(x_a e^{-\tau} G_a) \exp(te^{-2\tau} K) e^{\varphi_+ E_+} e^{-\tau D}, \\
e^{-\tau H} g_+ &= e^{x_0 G_0} e^{x_+ e^{\tau} G_+} e^{x_- e^{-\tau} G_-} e^{\varphi_+ e^{-\tau} E_+} e^{-\tau H}.
\end{aligned} \tag{2.126}$$

The above expressions yield the following representation by the left action:

$$\begin{aligned}\pi_L(M) &= -\Lambda(M), \\ \pi_L(D) &= -\Lambda(D) - \sum_{a=1}^3 x_a \frac{\partial}{\partial x_a} - 2t \frac{\partial}{\partial t}, \\ \pi_L(H) &= -\Lambda(H) + x_+ \frac{\partial}{\partial x_+} - x_- \frac{\partial}{\partial x_-} - \varphi_+ \frac{\partial}{\partial \varphi_+}.\end{aligned}\quad (2.127)$$

– Action of $\hat{s}(3)^+$

$$\begin{aligned}e^{-\tau G_0} g_+ &= e^{(x_0 - \tau) G_0} e^{x_+ G_+} e^{x_- G_-} e^{tK} e^{\varphi_+ E_+}, \\ e^{-\tau G_+} g_+ &= e^{x_0 G_0} e^{(x_+ - \tau) G_+} e^{x_- G_-} e^{tK} e^{\varphi_+ E_+}, \\ e^{-\tau G_-} g_+ &= e^{x_0 G_0} e^{x_+ G_+} e^{(x_- - \tau) G_-} e^{tK} e^{\varphi_+ E_+}, \\ e^{-\tau K} g_+ &= e^{x_0 G_0} e^{x_+ G_+} e^{x_- G_-} e^{(t - \tau) K} e^{\varphi_+ E_+}, \\ e^{-\tau E_+} g_+ &= e^{(x_0 + \tau x_+) G_0} e^{x_+ G_+} e^{(x_- - \tau x_0) G_-} e^{tK} e^{(\varphi_+ - \tau) E_+},\end{aligned}\quad (2.128)$$

yielding

$$\begin{aligned}\pi_L(G_0) &= -\frac{\partial}{\partial x_0}, \quad \pi_L(G_+) = -\frac{\partial}{\partial x_+}, \quad \pi_L(G_-) = -\frac{\partial}{\partial x_-}, \\ \pi_L(K) &= -\frac{\partial}{\partial t}, \quad \pi_L(E_+) = x_+ \frac{\partial}{\partial x_0} - x_0 \frac{\partial}{\partial x_-} - \frac{\partial}{\partial \varphi_+}.\end{aligned}\quad (2.129)$$

– Action of $\hat{s}(3)^-$:

$$\begin{aligned}\pi_L(P_0) &= -\Lambda(M)x_0 - t \frac{\partial}{\partial x_0}, \\ \pi_L(P_+) &= -\Lambda(M)x_+ - t \frac{\partial}{\partial x_+}, \\ \pi_L(P_-) &= -\Lambda(M)x_- - t \frac{\partial}{\partial x_-}, \\ \pi_L(P_t) &= -\Lambda(D)t - \frac{1}{2}\Lambda(M)(x_0^2 + x_+^2 + x_-^2) - t \sum_{a=1}^3 x_a \frac{\partial}{\partial x_a} - t^2 \frac{\partial}{\partial t}, \\ \pi_L(E_-) &= -\Lambda(H)\varphi_+ + x_- \frac{\partial}{\partial x_0} - x_0 \frac{\partial}{\partial x_+} - \frac{1}{2}\varphi_+^2 \frac{\partial}{\partial \varphi_+}.\end{aligned}\quad (2.130)$$

Right action

– By definition

$$\pi_R(M) = \Lambda(M), \quad \pi_R(D) = \Lambda(D), \quad \pi_R(H) = \Lambda(H). \quad (2.131)$$

– The right action of $\hat{s}(3)^+$ is given by (again up to terms quadratic in τ):

$$\begin{aligned}g_+ e^{\tau G_0} &= e^{(x_0 + \tau) G_0} e^{x_+ G_+} e^{(x_- + \tau \varphi_+) G_-} e^{tK} e^{\varphi_+ E_+}, \\ g_+ e^{\tau G_+} &= e^{(x_0 - \tau \varphi_+) G_0} e^{x_+ G_+} e^{(x_- \tau \frac{1}{2} \varphi_+^2) G_-} e^{tK} e^{\varphi_+ E_+},\end{aligned}$$

$$\begin{aligned}
g_+ e^{\tau G_-} &= e^{x_0 G_0} e^{x_+ G_+} e^{(x_- + \tau) G_-} e^{tK} e^{\varphi_+ E_+}, \\
g_+ e^{\tau K} &= e^{x_0 G_0} e^{x_+ G_+} e^{x_- G_-} e^{(t + \tau K)} e^{\varphi_+ E_+}, \\
g_+ e^{\tau E_+} &= e^{x_0 G_0} e^{x_+ G_+} e^{x_- G_-} e^{tK} e^{(\varphi_+ + \tau) E_+},
\end{aligned} \tag{2.132}$$

which yields

$$\begin{aligned}
\pi_R(G_0) &= \frac{\partial}{\partial x_0} + \varphi_+ \frac{\partial}{\partial x_-}, \\
\pi_R(G_+) &= \frac{\partial}{\partial x_+} - \varphi_+ \frac{\partial}{\partial x_0} - \frac{1}{2} \varphi_+^2 \frac{\partial}{\partial x_-}, \\
\pi_R(G_-) &= \frac{\partial}{\partial x_-}, \quad \pi_R(K) = \frac{\partial}{\partial t}, \quad \pi_R(E_+) = \frac{\partial}{\partial \varphi_+}.
\end{aligned} \tag{2.133}$$

2.3.2 Singular vectors

In this case a basis in the Verma module $V^\Lambda \cong U(\hat{\mathfrak{s}}(3)^+) v_0$ is

$$v_{\omega\alpha\beta\gamma\sigma} = G_0^\omega G_+^\alpha G_-^\beta K^\gamma E_+^\sigma v_0, \tag{2.134}$$

where $\omega, \alpha, \beta, \gamma, \sigma$ are nonnegative integers. Using the properties of the lowest weight vector

$$\begin{aligned}
M v_0 &= \Lambda(M) v_0 = M v_0, \\
D v_0 &= \Lambda(D) v_0 = \Delta v_0, \\
H v_0 &= \Lambda(H) v_0 = -h v_0,
\end{aligned} \tag{2.135}$$

we first calculate the action of the Cartan generators on the basis (2.134):

$$\begin{aligned}
M v_{\omega\alpha\beta\gamma\sigma} &= \Lambda(M) v_{\omega\alpha\beta\gamma\sigma}, \\
D v_{\omega\alpha\beta\gamma\sigma} &= (p + \Delta) v_{\omega\alpha\beta\gamma\sigma}, \quad p = \omega + \alpha + \beta + 2\gamma, \\
H v_{\omega\alpha\beta\gamma\sigma} &= (r - h) v_{\omega\alpha\beta\gamma\sigma}, \quad r = -\alpha + \beta + \sigma.
\end{aligned} \tag{2.136}$$

In the same manner we calculate the action of generators from $\hat{\mathfrak{s}}(3)^-$

$$\begin{aligned}
P_0 v_{\omega\alpha\beta\gamma\sigma} &= 2\omega \Lambda(M) v_{\omega-1, \alpha\beta\gamma\sigma} + \gamma v_{\omega+1, \alpha\beta, \gamma-1, \sigma}, \\
P_+ v_{\omega\alpha\beta\gamma\sigma} &= 2\beta \Lambda(M) v_{\omega, \alpha, \beta-1, \gamma\sigma} + \gamma v_{\omega, \alpha+1, \beta, \gamma-1, \sigma}, \\
P_- v_{\omega\alpha\beta\gamma\sigma} &= 2\alpha \Lambda(M) v_{\omega, \alpha-1, \beta\gamma\sigma} + \gamma v_{\omega, \alpha, \beta+1, \gamma-1, \sigma}, \\
P_t v_{\omega\alpha\beta\gamma\sigma} &= \omega(\omega-1) \Lambda(M) v_{\omega-1, \alpha, \beta, \gamma\sigma} + 2\alpha\beta \Lambda(M) v_{\omega, \alpha-1, \beta-1, \gamma, \sigma} \\
&\quad + \gamma(p - \gamma + \Delta - 1) v_{\omega\alpha\beta, \gamma-1, \sigma}, \\
E_- v_{\omega\alpha\beta\gamma\sigma} &= \omega v_{\omega-1, \alpha+1, \beta\gamma\sigma} - \beta v_{\omega+1, \alpha, \beta-1, \gamma, \sigma} - \frac{1}{2} \sigma(\sigma-1+2h) v_{\omega\alpha\beta\gamma\sigma-1}.
\end{aligned} \tag{2.137}$$

Next we apply the requirement $v_{\omega\alpha\beta\gamma\sigma}$ to be homogeneous, from which follows that r and p are to be fixed. Thus, from the five variables $\omega, \alpha, \beta, \gamma, \sigma$ only three are independent. We choose ω, α, σ as independent variables, which means that the singular vectors must have the form

$$v_s = \sum_{\omega, \alpha, \sigma} a_{\omega\alpha\sigma} v_{\omega\alpha\beta\gamma\sigma}, \quad (2.138)$$

the sum obeying the following:

The replacement $\alpha \rightarrow \alpha \pm 1$ causes the change of dependent variables

$$\beta \rightarrow \beta \pm 1, \quad \gamma \rightarrow \gamma \mp 1.$$

The replacement $\omega \rightarrow \omega \pm 2$ causes the change $\gamma \rightarrow \gamma \mp 1$ and does not reflect on the others.

Next we use the definition of a singular vector

$$Hv_s = \Lambda'(H)v_s, \quad H \in \hat{\mathfrak{s}}(3)^0; \quad Xv_s = 0; \quad X \in \hat{\mathfrak{s}}(3)^-, \quad (2.139)$$

to obtain

$$1. \quad P_0 v_s = 0$$

$$\begin{aligned} & 2\omega\Lambda(M)a_{\omega\alpha\sigma}v_{\omega-1,\alpha\beta\sigma} + \gamma a_{\omega\alpha\sigma}v_{\omega+1,\alpha\beta,\gamma-1,\sigma} \\ & = (2\omega\Lambda(M)a_{\omega\alpha\sigma} + (\gamma+1)a_{\omega-2,\alpha\sigma})v_{\omega-1,\alpha\beta\gamma\sigma} = 0 \\ & a_{\omega\alpha\sigma} = -\frac{\gamma+1}{2\Lambda(M)\omega}a_{\omega-2,\alpha\sigma}. \end{aligned} \quad (2.140)$$

$$2. \quad P_+ v_s = 0$$

$$a_{\omega\alpha\sigma} = -\frac{\gamma+1}{2\Lambda(M)\beta}a_{\omega,\alpha-1,\sigma}. \quad (2.141)$$

$$3. \quad P_- v_s = 0$$

$$a_{\omega\alpha\sigma} = -\frac{\gamma+1}{2\Lambda(M)\alpha}a_{\omega,\alpha-1,\sigma}. \quad (2.142)$$

$$4. \quad P_t v_s = 0$$

$$\begin{aligned} & (\omega+1)(\omega+2)\Lambda(M)a_{\omega+2,\alpha\sigma} + 2(\alpha+1)^2\Lambda(M)a_{\omega,\alpha+11,\beta-1,\sigma} \\ & + \gamma(p-\gamma+\Delta-1)a_{\omega\alpha\sigma} = 0. \end{aligned} \quad (2.143)$$

$$5. \quad E_- v_s = 0$$

$$(\omega a_{\omega,\alpha-1,\sigma} - \alpha a_{\omega-2,\alpha,\sigma})v_{\omega-1,\alpha,\beta-1,\gamma+1,\sigma} + \frac{1}{2}\sigma(\sigma-1-2h)a_{\omega\alpha\sigma}v_{\omega\alpha\beta\gamma\sigma-1} = 0, \quad (2.144)$$

We note that from (2.140) the substitution $\omega = 1$ leads to $a_{\omega\alpha\sigma} = 0$ for ω odd, while from (2.141) and (2.142) it follows that $\alpha = \beta$. Further substitution of (2.140) and (2.141) in (2.143) gives

$$\begin{aligned} \gamma(p/2 + \Delta - 5/2) &= 0, \\ p &= \omega + 2\alpha + 2\gamma = (2\Delta + 5)/2. \end{aligned} \quad (2.145)$$

Then p must also be even.

Solving the above recurrence relations was done in [8]. The result for the most general form of the singular vector is

$$v_s^{p,\sigma} = c \left(\frac{1}{2} G_0^2 + G_+ G_- - 2\Lambda(M)K \right)^{p/2} E_+^\sigma v_0, \quad (2.146)$$

where σ has two values: $\sigma = 0, 1 + 2h \in \mathbb{N}$, $p = 5 - 2\Delta \in 2\mathbb{N}$, c arbitrary nonzero constant.

Furthermore, one can verify that $v_s^\sigma = E_+^\sigma v_0 = v_s^{0,\sigma}$ is also a singular vector. In fact the relations

$$P_0 v_s^\sigma = P_+ v_s^\sigma = P_- v_s^\sigma = P_t v_s^\sigma = 0 \quad (2.147)$$

are automatically satisfied. The last condition,

$$E_- v_s^\sigma = \frac{1}{2} \sigma(\sigma - 1 - 2h) v_s^\sigma = 0, \quad (2.148)$$

is satisfied again for $\sigma = 1 + 2h = 1 - 2\Lambda(H)$.

Thus, $v_s^{p,\sigma}$ is a composite singular vector and we have a quartet to Verma modules with weights $\Lambda = (\Lambda(M) = M, \Lambda(D) = \Delta, \Lambda(H) = -h)$, $\Lambda_p = (M, p + \Delta, -h)$, $\Lambda'_\sigma = (M, \Delta, \sigma - h)$, $\Lambda_{p,\sigma} = (M, p + \Delta, \sigma - h)$, which can be given in an embedding commutative diagram:

$$\begin{array}{ccc} V^\Lambda & \longrightarrow & V^{\Lambda_p} \\ \downarrow & & \downarrow \\ V^{\Lambda'_\sigma} & \longrightarrow & V^{\Lambda_{p,\sigma}} \end{array} \quad (2.149)$$

where each arrow points to the embedded module. By construction, V^Λ has singular vectors $v_s^{p,0}$ and $v_s^{0,\sigma}$, $V^{\Lambda'_\sigma}$ has singular vector $v_s^{p,0}$, V^{Λ_p} has singular vector $v_s^{0,\sigma}$.

All above properties follow from the following relations:

$$\begin{aligned} M v_s^{p,\sigma} &= \Lambda(M) v_s^{p,\sigma}, \\ D v_s^{p,\sigma} &= (\Lambda(D) + p) v_s^{p,\sigma}, \\ H v_s^{p,\sigma} &= (\Lambda(H) + \sigma) v_s^{p,\sigma}. \end{aligned} \quad (2.150)$$

2.3.3 Non-relativistic equations

Equation (2.146) yields (taking the right action) the following form of invariant differential operator and correspondingly invariant equation:

$$\begin{aligned}\mathcal{D}_{p,\sigma} &= \left(\sum_{i=1}^3 \frac{\partial^2}{\partial x_i^2} + 2M\partial_t \right)^{p/2} (\partial_{\varphi_+})^\sigma, \\ \mathcal{D}_{p,\sigma}\psi(t, x_1, x_2, x_3; \varphi_+) &= \psi'(t, x_1, x_2, x_3; \varphi_+),\end{aligned}\quad (2.151)$$

where ψ belongs to the representation C^Λ characterized by Λ , while ψ' belongs to the representation $C^{\Lambda_{p,\sigma}}$ characterized by $\Lambda_{p,\sigma}$.

Of course, analogously to the Verma module embedding picture (2.149) there is a quartet commutative diagram:

$$\begin{array}{ccc} C^\Lambda & \longrightarrow & C^{\Lambda_p} \\ \downarrow & & \downarrow \\ C^{\Lambda'_\sigma} & \longrightarrow & C^{\Lambda_{p,\sigma}} \end{array}\quad (2.152)$$

where each vertical arrow depicts the differential operator $\mathcal{D}_{0,\sigma} = (\partial_{\varphi_+})^\sigma$, while each horizontal arrow depicts the differential operator $\mathcal{D}_{p,0}$.

2.4 q -Schrödinger algebra

2.4.1 q -deformation of the Schrödinger algebra

In this section we review [176]. For other approaches to q -deformations of the Schrödinger algebra we refer, e. g. to [233, 102, 36, 4, 373, 374].

We use the following q -number notations:

$$[a]_q \doteq \frac{q^a - q^{-a}}{q - q^{-1}}, \quad [a]'_q \doteq [a]_{q^{1/2}} = \frac{q^{a/2} - q^{-a/2}}{q^{1/2} - q^{-1/2}} = \frac{[a/2]_q}{[1/2]_q}, \quad (2.153)$$

and similarly for any diagonal operator instead of a .

Now we construct a q -deformation $\hat{S}_q(1)$ of the Schrödinger algebra under the following conditions:

1. A realization of the generators P_t, P, G, K in terms of q -difference operators and multiplication operators should be available.
2. In the limit $q \rightarrow 1$ we should have the classical relations (1.189).
3. The subalgebra structure should be preserved by the deformation and especially the q -deformed $\mathfrak{sl}(2, \mathbb{C})$ subalgebra generated by D, K, P_t should coincide with the usual Drinfeld–Jimbo deformation [196, 342] $U_q(\mathfrak{sl}(2, \mathbb{C}))$.

With these conditions we get for $\hat{S}_q(1)$ the following nontrivial relations instead of (1.189):

$$P_t G - q G P_t = P_x, \quad (2.154a)$$

$$[P_x, K] = G q^{-D}, \quad (2.154b)$$

$$[D, G] = G, \quad (2.154c)$$

$$[D, P_x] = -P_x, \quad (2.154d)$$

$$[D, P_t] = -2P_t, \quad (2.154e)$$

$$[D, K] = 2K, \quad (2.154f)$$

$$[P_t, K] = [D]_q \quad (2.154g)$$

$$P_x G - q^{-1} G P_x = m, \quad (2.154h)$$

$$P_t P_x - q^{-1} P_x P_t = 0. \quad (2.154i)$$

Conditions 2. and 3. can now be checked directly; (2.154e,f,g) are the standard commutation relations of the Drinfeld–Jimbo deformation $U_q(\mathfrak{sl}(2, \mathbb{C}))$. Moreover, we obtain a q -deformed centrally extended Galilei subalgebra generated by P_t, P, G . The deformation is a “mild” one, in the sense that commutators are turned into q -commutators, cf. (2.154a,h,i), and it differs from the Galilei algebra q -deformation given in [76], which is not a surprise taking into account that the latter is not a subalgebra of a (q -deformed) Schrödinger algebra. Condition 1. will be discussed later.

The commutation relations (2.154) are graded as the undeformed ones (1.189) if we define the grading as in (1.178).

For *real* q the commutation relations (2.154) are preserved also by the involutive antiautomorphism (1.179) supplemented by the condition $\omega(q) = q$. With this conjugation the subalgebra generated by $P_t + K, i(P_t - K), D$ (the fixed points of ω) is $U_q(\mathfrak{sl}(2, \mathbb{R}))$.

2.4.2 Lowest weight modules of $\hat{S}_q(1)$

We consider lowest weight modules (LWM) of $\hat{S}_q(1)$, in particular, Verma modules, in complete analogy with the undeformed case $q = 1$. In particular, the lowest weight vector fulfils (2.4). The Verma module V^Δ is given explicitly by $V^\Delta = U_q(S^+) \otimes v_0$, where $U_q(S^+)$ is the q -deformed universal enveloping algebra of $S^+ \equiv S(1)^+$. Clearly, $U_q(S^+)$ has the basis elements $p_{k,\ell} = G^k K^\ell$. The basis vectors of the Verma module are $v_{k,\ell} = p_{k,\ell} \otimes v_0$ (with $v_{0,0} = v_0$). The action of the q -Schrödinger algebra on this basis is derived easily from (2.154):

$$D v_{k,\ell} = (k + 2\ell + \Delta) v_{k,\ell}, \quad (2.155a)$$

$$G v_{k,\ell} = v_{k+1,\ell}, \quad (2.155b)$$

$$K v_{k,\ell} = v_{k,\ell+1}, \quad (2.155c)$$

$$P_x v_{k\ell} = q^{\frac{1-k}{2}} M[k]_q' v_{k-1,\ell} + q^{1-\Delta-\ell-k} [\ell]_q v_{k+1,\ell-1}, \quad (2.155d)$$

$$P_t v_{k\ell} = [\ell]_q [k + \ell - 1 + \Delta]_q v_{k,\ell-1} + M \frac{[k]_q' [k-1]_q'}{[2]_q'} v_{k-2,\ell}. \quad (2.155e)$$

For the derivation of (2.155) the following relations (which follow from (2.154)) are useful:

$$P_x G^k - q^{-k} G^k P_x = M q^{(1-k)/2} [k]_q' G^{k-1}, \quad (2.156a)$$

$$P_x K^\ell - K^\ell P_x = q^{1-\ell} [\ell]_q G K^{\ell-1} q^{-D}, \quad (2.156b)$$

$$P_t G^k - q^k G^k P_t = [k]_q G^{k-1} P_x + \frac{[k]_q' [k-1]_q'}{[2]_q'} G^{k-2}, \quad (2.156c)$$

$$P_t K^\ell - K^\ell P_t = [\ell]_q K^{\ell-1} [D + \ell - 1]_q. \quad (2.156d)$$

Because of (2.155)a we notice that the Verma module V^Δ can be decomposed in homogeneous subspaces w. r. t. the grading operator D as in (2.6).

Next we analyze the reducibility of V^Δ through singular vectors. The considerations are exactly similar to the undeformed case. All possible singular vectors were given explicitly in [176] as follows. Fix the grade $p > 0$ and denote the singular vector as v_s^p . Consider the case of *even* grade, $p \in 2\mathbb{N}$. Since $v_s^p \in V_p^\Delta$ we have

$$v_s^p = \sum_{\ell=0}^{p/2} a_\ell v_{p-2\ell,\ell} = \mathcal{Q}^p(G, K) \otimes v_0, \quad p \text{ even}. \quad (2.157)$$

Applying (2.7) we see that a singular vector exists only for $\Delta = \frac{3-p}{2}$ (as for $q = 1$) and is given explicitly for arbitrary q by the formula

$$v_s^p = a_0 \sum_{\ell=0}^{\frac{p}{2}} (-M[2]_q')^\ell \binom{\frac{p}{2}}{\ell}_q v_{p-2\ell,\ell} = a_0 (G^2 - M[2]_q' K)_q^{\frac{p}{2}} \otimes v_0, \\ \mathcal{Q}^p(G, K) = a_0 (G^2 - M[2]_q' K)_q^{\frac{p}{2}}, \quad (2.158)$$

where

$$\binom{p}{s}_q \doteq \frac{[p]_q!}{[s]_q! [p-s]_q!}, \quad [n]_q! \doteq [n]_q [n-1]_q \dots [1]_q. \quad (2.159)$$

For *odd* grade there are no singular vectors as for $q = 1$.

To analyze the consequences of the reducibility of our Verma modules we take the subspace of $V^{(3-p)/2}$:

$$I^{(3-p)/2} = U_q(S^+) v_s^p. \quad (2.160)$$

It is invariant under the action of the q -deformed Schrödinger algebra, and is isomorphic to a Verma module $V^{\Delta'}$ with shifted weight $\Delta' = \Delta + p = (p+3)/2$. The latter Verma module has no singular vectors.

Let us denote by $\mathcal{L}^{(3-p)/2}$ the factor-module $V^{(3-p)/2}/I^{(3-p)/2}$ and by $|p\rangle$ the lowest weight vector of $\mathcal{L}^{(3-p)/2}$. As a consequence of (2.7) and (2.158) $|p\rangle$ satisfies

$$P_x|p\rangle = 0, \quad (2.161a)$$

$$P_t|p\rangle = 0, \quad (2.161b)$$

$$\sum_{\ell=0}^{\frac{p}{2}} (-m[2]_q')^\ell \binom{\frac{p}{2}}{\ell}_q G^{p-2\ell} K^\ell |p\rangle = 0. \quad (2.161c)$$

Now from (2.161c) we see that

$$K^{p/2}|p\rangle = - \sum_{\ell=0}^{p/2-1} \frac{1}{(-m[2]_q')^{p/2-\ell}} \binom{p/2}{\ell}_q G^{p-2\ell} K^\ell |p\rangle. \quad (2.162)$$

By repeated application of this relation to the basis one can get rid of all powers $\geq p/2$ of K . Thus the basis of $\mathcal{L}^{(3-p)/2}$ will be a *singleton basis* for $p = 2$, and a *quasi-singleton basis* for $p \geq 4$:

$$\dim V_n^{(3-p)/2} = 1, \quad \text{for } n = 0, 1 \text{ or } n \geq p, \quad (2.163)$$

and it is given by

$$v_{k\ell}^p \equiv G^k K^\ell |p\rangle, \quad p \in 2\mathbb{N}, \quad k, \ell \in \mathbb{Z}_+, \quad \ell \leq p/2 - 1, \quad \Delta = \frac{3-p}{2}. \quad (2.164)$$

The transformation rules of this basis are (2.155) except (2.155c) for $\ell = p/2 - 1$, when we have

$$Kv_{k,p/2-1}^p = - \sum_{s=0}^{p/2-1} \frac{1}{(-m[2]_q')^{p/2-s}} \binom{p/2}{s}_q v_{k+p-2s,s}^p. \quad (2.155c')$$

From the transformation rules we see that $\mathcal{L}^{(3-p)/2}$ is irreducible. In the simplest case $p = 2$ the irrep $\mathcal{L}^{1/2}$ is also an irrep of the q -deformed centrally extended Galilean subalgebra $\hat{\mathcal{G}}_q(1)$ generated by P_x, P_t, G .

Hence, the complete list of the irreducible lowest weight modules over the q -deformed centrally extended Schrödinger algebra is given by [176]

- V^Δ , when $d \neq (3-p)/2$, $p \in 2\mathbb{N}$;
- $\mathcal{L}^{(3-p)/2}$, when $\Delta = (3-p)/2$, $p \in 2\mathbb{N}$.

These irreps are infinite-dimensional.

2.4.3 Vector-field realization of $\hat{S}_q(1)$ and generalized q -deformed heat equations

Let us introduce the “number” operator N_y for the coordinate $y = x, t$, i. e.

$$N_y y^k = k y^k, \quad (2.165)$$

and the q -difference operators $\mathcal{D}_y, \mathcal{D}'_y$, which admit a general definition on a larger domain than polynomials; but on polynomials they are well defined as follows:

$$\mathcal{D}_y \doteq \frac{1}{y} [N_y]_q, \quad (2.166a)$$

$$\mathcal{D}'_y \doteq \frac{1}{y[\frac{1}{2}]_q} \left[\frac{N_y}{2} \right]_q = \frac{1}{y} [N_y]_q', \quad (2.166b)$$

so that for any suitable function f we obtain as a consequence of (2.165)

$$\mathcal{D}_y f(y) = \frac{f(qy) - f(q^{-1}y)}{y(q - q^{-1})}, \quad (2.167a)$$

$$\mathcal{D}'_y f(y) = \frac{f(q^{\frac{1}{2}}y) - f(q^{-\frac{1}{2}}y)}{y(q^{\frac{1}{2}} - q^{-\frac{1}{2}})}. \quad (2.167b)$$

For $q \rightarrow 1$ one has: $N_y \rightarrow y\partial_y$, $\mathcal{D}_y, \mathcal{D}'_y \rightarrow \partial_y$.

With this notation there exists [176] a five-parameter realization of (2.154) via q -difference operators:

$$P_t = q^{c_1} \mathcal{D}_t q^{(1-c_5)N_t + (1-c_4)N_x}, \quad (2.168a)$$

$$P_x = q^{c_2} \mathcal{D}'_x q^{-c_4 N_t + (c_3 + \frac{1}{2})N_x}, \quad (2.168b)$$

$$D = 2N_t + N_x + \Delta, \quad (2.168c)$$

$$G = q^{c_2 - c_1 - c_4 + c_5} t \mathcal{D}'_x q^{(c_5 - c_4)N_t + (c_3 + c_4 - \frac{1}{2})N_x} \\ + q^{-c_2 - c_3 - \frac{1}{2}} m x q^{c_4 N_t - (c_3 + 1)N_x}, \quad (2.168d)$$

$$K = q^{-c_1 + c_5 - 1 - \Delta} t^2 \mathcal{D}_t q^{(c_5 - 1)N_t + c_4 N_x} \\ + q^{-c_1 + c_5 - 1 - \Delta} t x \mathcal{D}_x q^{(c_5 - 2)N_t + (c_4 - 1)N_x} \\ + q^{-c_1 + c_5 - 1} [\Delta] q^{c_5 N_t + c_4 N_x} \\ + q^{-2c_2 - 3c_3 - \frac{3}{2} - \Delta} \left[\frac{1}{2} \right]_q m x^2 q^{2(c_4 - 1)N_t - 2(c_3 + 1)N_x}, \quad (2.168e)$$

where c_1, c_2, c_3, c_4, c_5 are arbitrary parameters. (There might be other realizations that are not equivalent to the one just given.)

For $q = 1$ we recover the standard vector-field realization of $\hat{S}(1)$ (1.195).

Our realization (2.168) may be used to construct a polynomial realization of the irreducible lowest weight modules considered in Section 2.3. For that case we represent

the lowest weight vector by the function 1. Indeed, the constants in (2.168) are chosen so that (2.4) is satisfied:

$$D1 = \Delta, \quad P_x 1 = 0, \quad P_t 1 = 0. \quad (2.169)$$

Applying the basis elements $p_{k,\ell} = G^k K^\ell$ of the universal enveloping algebra $U_q(\mathcal{S}^+)$ to 1 we get polynomials in x, t which will be denoted by $f_{k,\ell} \equiv p_{k,\ell} 1$. For the explicit expressions we refer to [176]. There it was also shown that the basis $f_{k,\ell}$ is a realization of the irreducible lowest weight representations of $\hat{S}_q(1)$ listed at the end of the previous section. Indeed, there is 1-to-1 correspondence between the states $v_{k,\ell}$ of the Verma modules over $\hat{S}_q(1)$ and the polynomials $f_{k,\ell}$. The irreducible lowest weight representations of $\hat{S}_q(1)$ are factor-modules of Verma modules, with factorization over the invariant subspaces generated by singular vectors. This statement is trivial if there is no singular vector. When a singular vector exists, i. e. for the representations $V^{(3-p)/2}$, we first obtain a q -difference operator by substituting in $\mathcal{Q}^p(G, K)$ (cf. (2.157) and (2.158)) each generator with its vector-field realization. For the irreducibility of $\mathcal{L}^{(3-p)/2}$ it is enough to show that the q -difference operator $\mathcal{Q}^p(G, K)$ vanishes identically when applied to 1. This contains more information, as $\mathcal{Q}^p(G, K)$ gives also a q -difference equation invariant under the action of $\hat{S}_q(1)$. Because of this invariance the solutions of this equation are elements of $\mathcal{L}^{(3-p)/2}$. Thus we have an infinite family of q -difference equations, the family members being labelled by $p \in 2\mathbb{N}$, i. e. we have one equation for each representation space $V^{(3-p)/2}$. These equations may be called generalized q -deformed heat equations (m real) or generalized q -deformed Schrödinger equations (m imaginary). The case $p = 2$ is a q -difference analog of the ordinary heat/Schrödinger equation.

Before presenting the last example explicit we make a choice of constants in (2.168) and set for simplicity $c_1 = c_2 = c_3 = c_4 = c_5 = 0$ so as to work with simpler expressions for the generators:

$$P_t = \mathcal{D}_t q^{N_t + N_x}, \quad (2.170a)$$

$$P_x = \mathcal{D}_x' q^{\frac{1}{2}N_x}, \quad (2.170b)$$

$$D = 2N_t + N_x + \Delta, \quad (2.170c)$$

$$G = t\mathcal{D}_x' q^{-\frac{1}{2}N_x} + q^{-\frac{1}{2}} Mx q^{-N_x}, \quad (2.170d)$$

$$K = q^{-\Delta-1} t^2 \mathcal{D}_t q^{-N_t} + q^{-\Delta-1} t x \mathcal{D}_x q^{-2N_t - N_x} \\ + q^{-1} [\Delta]_q t + q^{-\Delta-\frac{3}{2}} \left[\frac{1}{2} \right]_q Mx^2 q^{-2N_t - 2N_x}. \quad (2.170e)$$

The operator $S_q = \mathcal{Q} = G^2 - [2]_q' MK$ determining the singular vectors reads

$$S_q = t^2 q^{\frac{1}{2}} (\mathcal{D}_x'^2 q^{-N_x} - q^{-\Delta-\frac{3}{2}} [2]_q' M \mathcal{D}_t q^{-N_t}) \\ + M t x \mathcal{D}_x' ([2]_q' - (1 + q^{N_x}) q^{-\Delta-1-2N_t}) q^{-\frac{3}{2}N_x} \\ + q^{-1} M t (q^{-2N_x} - [2\Delta]_q')$$

$$+ q^{-2} M^2 x^2 (1 - q^{-\Delta + \frac{1}{2} - 2N_t}) q^{-2N_x}, \quad (2.171)$$

which for $q = 1$ gives

$$S = t^2 (\partial_x^2 - 2M\partial_t) + Mt(1 - 2\Delta). \quad (2.172)$$

Hence we interpret for $\Delta = \frac{1}{2}$ (which corresponds to the lowest singular vector ($p = 2$)) the equation $S_q f = 0$ as a *q-deformed heat/Schrödinger equation* as we motivated in Section 2.1. The explicit form of this equation is

$$\begin{aligned} S_q f &= 0 \\ S_q &= t^2 q^{\frac{1}{2}} (\mathcal{D}'_x{}^2 q^{-N_x} - q^{-2} [2]_q' M \mathcal{D}_t q^{-N_t}) \\ &\quad + M t x \mathcal{D}'_x ([2]_q' - (1 + q^{N_x}) q^{-\frac{3}{2} - 2N_t}) q^{-\frac{3}{2} N_x} \\ &\quad - \lambda q^{-1} M t x \mathcal{D}_x q^{-N_x} + \lambda q^{-2} M^2 x^2 t \mathcal{D}_t q^{-N_t - 2N_x} \end{aligned} \quad (2.173)$$

where $\lambda \doteq q - q^{-1}$.

This is the proposal of [176] for a *q-deformed heat equation*. For $q \mapsto 1$ ($\lambda \mapsto 0$) this equation leads to the ordinary heat/Schrödinger equation.

Remark 4. We note that there exists another *q*-deformation of the vector-field realization of $\hat{S}(1)$ given in [233]. They start with a special *q*-deformed heat equation and look for a *q*-symmetry algebra on its solution variety. The resulting *q*-deformation of the Schrödinger algebra in [233], which we call an *on shell* deformation is different from the one of [176] and is valid only on the solutions of the *q*-deformed heat equation under consideration. $\diamond$

2.5 Difference analogues of the free Schrödinger equation

2.5.1 Motivations

In this section we review [177]. For other approaches to difference equations with Schrödinger algebra symmetry we refer, e. g. to [233, 417].

The time evolution of physical systems is generically described via (partial) differential equations, especially via the Schrödinger equation for the case of non-relativistic quantum mechanics. The use of such equations, however, is an idealization, because the infinitesimal structure inherent in the definition of differential operators,

$$\partial_x f(x_0) \equiv \lim_{\zeta \rightarrow 0} \frac{f(x_0 + \zeta) - f(x_0 - \zeta)}{2\zeta}, \quad (2.174)$$

cannot be reproduced in physical measurements. A *realistic* measurement of a “differential quantity” such as velocity etc. actually involves measurements at two *distinct*

points in the physical space-time, i. e. it is based on measurements at x and $x + \zeta$ with ζ *finite* and not infinitesimal. Hence, in a realistic physical setting, only finite difference quotients of non-infinitesimal quantities should occur. The physical space-time could be either continuous or discrete for the space and/or time coordinates. On very small length scales (Planck scale) it is likely that some “grained” or “lattice” structure is more appropriate as an “arena” for physical theories than a continuous space-time.

Apart from such more fundamental considerations there are also practical reasons for the use of finite difference equations. Generically, one has to use *approximations* in order to get a quantitative description of a physical system, and sometimes lattice models are useful in this context. In these models the objects of the theory are allowed to “live” only on discrete points of a lattice. It is obvious that in this setting finite difference operators involving the lattice spacing are basic quantities.

There are various types of (finite) difference operators. The usual choice, used also in the context of lattices, are operators of additive type:

$$D_x f(x) = \frac{f(x + \zeta) - f(x - \zeta)}{2\zeta}, \quad (2.175)$$

i. e. functions at $x \pm \zeta$ are compared. (Note that $\lim_{\zeta \rightarrow 0} D_x = \partial_x$.) Another type which was recently discussed in the context of generalized symmetries are difference operators of multiplicative type, the so-called q -difference operators:

$$D_x^q f(x) = \frac{f(qx) - f(q^{-1}x)}{x(q - q^{-1})} \quad (2.176)$$

which compare functions at qx and $q^{-1}x$. They appear naturally in the representation theory of quantum groups or more generally for q -deformed symmetries as we discussed in the previous section.

If one wants to model physical systems through difference operators one has to derive or to motivate, e. g. evolution equations as difference equations. A formal attempt uses of a kind of correspondence principle. This means to replace the usual differential operator by a difference operator so that in the continuum limit the “usual” theory is reproduced.

A more generic method is the following: If a differential equation or relation is derived from first principles, e. g. from a symmetry group or an algebra, one can try to formulate already the principles in terms of difference operators. If necessary, one has to change the derivation and possibly also further assumptions are needed. Certainly also here we will get in the limiting case a differential equation, however, the result may differ from the one obtained from the correspondence principle.

Here we discuss the free quantum mechanical Schrödinger equation (SE) without spin on $\mathbb{R}_x^n \times \mathbb{R}_t$ based on first principles: The SE is physically characterized through representation theory of the central extension of the $(n + 1)$ -dimensional Schrödinger algebra $\hat{S}$. We gave in previous sections a purely algebraic construction for the family of $\hat{S}$ invariant Schrödinger equations from a family of singular vectors in Verma

modules over $\hat{S}$. (We used also q -difference operators to derive a q -analogue of the Schrödinger equation, cf. [176] and the previous section.) Now, we use the general method and in the realization of $\hat{S}$ through vector fields we replace the vector fields with additive difference vector fields, i. e. vector fields with difference operators instead of differential operators. Because the construction of Verma modules and the construction of invariant differential equation is completely algebraic we can apply this method also in this case.

To relate the Schrödinger algebra invariance as a first principle for a quantum mechanical evolution equation with difference operators we need the definitions and some properties of these operators and a realization of $\hat{S}$ through difference vector fields. This construction is not unique: additional assumptions which are physically well motivated are necessary. The essential point in our derivation is the observation made in [175] (and in previous Sections) that the construction of invariant equations from Verma modules is independent of the realization of the generators of the corresponding algebra. The reason for this is that the construction of [175] is completely algebraic, i. e. it uses only the commutation relations of the algebra.

2.5.2 Definition and notation

As we discussed above the natural candidate for the formulation of difference analogues of differential equations are additive finite difference operators as in (2.175). In the representation of $\hat{S}$ one needs (partial) finite difference operators with respect to space-coordinates x_i and the time-coordinate t . We use τ as “fundamental (time-) length” for the time-coordinate and ξ as “fundamental (space-) length” for the space coordinate, i. e.

$$\begin{aligned} D_t f(t, x) &\equiv \frac{f(t + \tau, x) - f(t - \tau, x)}{2\tau}, \\ D_x f(t, x) &\equiv \frac{f(t, x + \xi) - f(t, x - \xi)}{2\xi}. \end{aligned} \quad (2.177)$$

The generalization to the $n + 1$ -dimensional case is obvious. Note that (2.175) is not the only possibility for the introduction of a finite difference operator with the “correct” limit; one could, e. g. use any expression of the form¹

$$D_z^{(a,b)} \doteq \frac{f(z + a\zeta) - f(z - b\zeta)}{(a + b)\zeta} \quad a, b \in \mathbb{Z}, \quad a \neq -b.$$

Thus, $D_z = D_z^{(1,1)}$.

¹ The requirement $a, b \in \mathbb{Z}$ reflects the fact that only entire multiples of the fundamental length ζ are considered measurable.

In the context of finite difference operators **shift operators** T_z^a , defined as

$$T_z^a f(z) \equiv f(z + a\zeta), \quad a \in \mathbb{Z}, \quad (2.178)$$

are useful. We have

$$D_z^{(a,b)} = \frac{T_z^a - T_z^{-b}}{(a+b)\zeta}.$$

Apart from the symmetric difference operator (2.175), we will also use a “forward difference operator”

$$D_z^+ \equiv D_z T_z = \frac{T_z^2 - 1}{2\zeta} \quad (2.179)$$

and a “backward difference operator”

$$D_z^- \equiv D_z T_z^{-1} = \frac{1 - T_z^{-2}}{2\zeta}. \quad (2.180)$$

For our purposes it is important that $D_z^{(a,b)}$ and T_z^a are, by definition, *linear operators*.

2.5.3 Construction of a realization of $\hat{\mathcal{S}}$

We construct a realization of $\hat{\mathcal{S}}$ with finite difference and shift operators. As explained above this is possible only if one imposes additional *assumptions* which we choose to be the following:

1. *In the zero-limit of the “fundamental lengths” the representation (1.176) should be recovered.*
2. *The number of additional terms with vanishing limit in the representation should be “as small as possible”.*
3. *The generators of space translations P_i (and of time translations P_t when we consider representations with difference operators in t) should take a “simple form”, i. e. they should be realized by the terms²*

$$P_i = D_i T_i^a, \quad a \in \mathbb{Z}. \quad (2.181)$$

It turns out that the following operators constitute a realization of $\hat{\mathcal{S}}$ obeying the above-mentioned assumptions [177]:

$$\begin{aligned} P_t &= D_t^+, \\ P_i &= D_i^+, \end{aligned} \quad (2.182)$$

² We use the shorthand notation $T_i \equiv T_{x_i}$ and $D_i \equiv D_{x_i}$.

$$\begin{aligned}
D &= 2tD_t^+ + \sum_{i=1}^n x_i D_i^- + \Delta, \\
J_{ij} &= x_j D_i^+ T_j^{-2} - x_i D_j^+ T_i^{-2}, \\
G_i &= tD_i^+ T_i^{-2} + Mx_i T_i^{-2}, \\
K &= (t^2 - 2\tau t)D_t^- T_t^{-2} + t \left(\sum_{i=1}^n x_i D_i^- \right) T_t^{-2} \\
&\quad + \frac{M}{2} \sum_{i=1}^n (x_i T_i^{-2})^2 + t\Delta T_t^{-2}.
\end{aligned}$$

By setting $\tau \rightarrow 0$, i. e.

$$D_t^\pm \rightarrow \partial_t, \quad T_t^a \rightarrow 1,$$

or $\xi \rightarrow 0$, i. e.

$$D_i^\pm \rightarrow \partial_i, \quad T_i^a \rightarrow 1,$$

one obtains realizations of $\hat{S}$ in which only space (respectively time) differentials are replaced by difference operators. We stress the fact that all these realizations of $\hat{S}$ are *linear*.

2.5.4 Invariant finite difference equations

Above we obtained $\hat{S}$ -invariant partial differential equations by inserting the $\hat{S}$ -realization (1.176) into the expression (2.8) which determines the singular vectors of the corresponding Verma modules. We mentioned already that the validity of this result does *not* depend on whether one has a realization with *differential* operators or not. In fact any *linear* realization leads to invariant equations. Thus, we can insert (2.182) into (2.8) and obtain $\hat{S}$ -invariant finite difference equations. As a result, we find that the equations

$$\left(D_t^+ - \frac{1}{2M} \left(\sum_{i=1}^n D_i^+ \right)^2 \right)^{\frac{p}{2}} \psi(t, x) = 0, \quad p \text{ even}, \quad (2.183)$$

are invariant under the $\hat{S}$ -realization (2.182) with $\Delta = \frac{n+2-p}{2}$.

As a special case ($p = 2$) we obtain a finite difference analogue of the free Schrödinger equation in n space dimensions:

$$\left(D_t^+ - \frac{1}{2M} \left(\sum_{i=1}^n D_i^+ \right)^2 \right) \psi(t, x) = 0. \quad (2.184)$$

Setting $\tau \rightarrow 0$ or $\xi \rightarrow 0$ in (2.182), respectively, we obtain discrete-continuous analogues of the free Schrödinger equation in which only space- or time- differentiation is replaced with the corresponding difference operators, i. e.

$$\left(\partial_t - \frac{1}{2M} \left(\sum_{i=1}^n D_i^+ \right)^2 \right) \psi(t, x) = 0, \quad (2.185)$$

$$\left(D_t^+ - \frac{1}{2M} \left(\sum_{i=1}^n \partial_i \right)^2 \right) \psi(t, x) = 0. \quad (2.186)$$

We stress that all these analogues of the free Schrödinger equation are *not* postulated but derived by our algebraic construction. Especially the appearance of D^+ —instead of another $D^{(a,b)}$ or linear combinations of several such operators—is forced by the assumptions given above and the construction of the equations.

In the case $n = 1$ equation (2.184) was considered in [233, 417], equations (2.185) and (2.186) in [417]. These authors looked for the symmetry of these equations employing difference or differential-difference operators, and they found that on the solution set of the equations these operators satisfy the $n = 1$ Schrödinger algebra, though the explicit expressions of the operators are different from ours (and for (2.184) between the two papers mentioned).

3 Virasoro algebra and super-Virasoro algebras

Summary

Starting with the fundamental paper [57] Belavin–Polyakov–Zamolodchikov (BPZ) there has been great interest in the two-dimensional conformal theories [245, 195]. Although the underlying Virasoro algebra was introduced a long time ago by Gelfand–Fuchs [266] in mathematics¹ and independently, by Virasoro [561] in physics, was extensively used in dual string theories (cf. e. g. [338]) the lack of a developed representation theory obviously hindered its further applications. It is no wonder that the BPZ paper appeared after the papers of Kac [352] and of Feigin–Fuchs [214]. Even a decade long work on the Liouville theory by Gervais–Neveu was better understood and reshaped (cf. [271], and also [376]). The above developments inevitably made contact with the Kac–Moody symmetry approach to current algebras [570, 134, 124]. That contact was initiated by Segal [533] who showed that every highest weight module (HWM) of a Kac–Moody algebra may be extended to the semi-direct product with the Virasoro algebra. Consequently, the elements of the latter can be realized as bilinear combinations (normally ordered) of elements of the Kac–Moody algebras (i. e. as elements of the latter universal enveloping algebras) [243, 533]. In the physical literature such constructions can be traced back to [548] but were reintroduced in the study of completely integrable models by Sato and others [528]. In the current algebra context they were mentioned by Polyakov [495] and used in full force in [392] (see also [277, 554]).

In this chapter, first, following [155] and [156] we give the multiplet classification of all reducible HWMs over the Virasoro and $N = 1$ super-Virasoro algebras. Along with a simple explicit parametrization of all reducible Verma modules, we show all the possible embedding maps between them. These embedding maps correspond one-to-one to all possible singular vectors of the reducible Verma modules. We recall that any singular vector becomes a homogeneous polynomial and, in a function space realization, it becomes a linear differential operator of an order equal to the degree of the corresponding polynomial. These degrees are given explicitly for the singular vectors corresponding to all noncomposition embedding maps and, thus, for all singular vectors. Further, following [157] we give the characters of all irreducible highest weight modules over the Virasoro algebra and $N = 1$ super-Virasoro algebras (Neveu–Schwarz superalgebra and Ramond superalgebra). These are given in an explicit and unified form for all three (super-)algebras incorporating all previously known results. Using the character formulae we introduce a Weyl group for Virasoro and $N = 1$ super-Virasoro algebras [164]. Further, following [109] we present formulae for singular vectors $c < 1$ Fock modules over the Virasoro algebras. These we present in terms of Schur polynomials generalizing the $c = 1$ expressions of Goldstone. We reproduce the known formulae for the singular vectors in $(1, n)$, $(m, 1)$ modules and give new formulae in the cases $(2, n)$, $(m, 2)$, $(3, 3)$. Furthermore, we present the characters of the unitarizable highest weight modules over the $N = 2$ superconformal algebras following mainly [79] and [158]. Finally, following [181] and [182] we consider modular invariants for theta-functions with characteristics and the twisted $N = 2$ superconformal and $\text{su}(2)$ Kac–Moody algebras and the classification of the corresponding modular invariant partition functions.

¹ Gelfand and Fuchs actually showed that the Witt algebra (a Lie algebra without a centre) has a unique one-dimensional central extension which was later called the Virasoro algebra.

3.1 Preliminaries on Virasoro and $N = 1$ super-Virasoro algebras

The *Virasoro algebra* [266, 561] $\widehat{W}$ is a complex Lie algebra with basis $\hat{c}, L_n, n \in \mathbb{Z}$ and Lie brackets:

$$[L_m, L_n] = (m - n)L_{m+n} + \delta_{m,-n} \frac{1}{12}(m^3 - m)\hat{c}, \quad (3.1a)$$

$$[\hat{c}, L_n] = 0. \quad (3.1b)$$

The *Neveu–Schwarz superalgebra* [470] $\widehat{S}$ is a complex Lie superalgebra with basis $\hat{c}, L_n, J_\alpha, n \in \mathbb{Z}, \alpha \in \mathbb{Z} + \frac{1}{2}$, and Lie (super-)brackets:

$$[L_m, L_n] = (m - n)L_{m+n} + \delta_{m,-n} \frac{1}{8}(m^3 - m)\hat{c}, \quad (3.2a)$$

$$[J_\alpha, J_\beta]_+ = 2L_{\alpha+\beta} + \delta_{\alpha,-\beta} \frac{1}{2}\left(\alpha^2 - \frac{1}{4}\right)\hat{c}, \quad (3.2b)$$

$$[L_m, J_\alpha] = \left(\frac{1}{2}m - \alpha\right)J_{m+\alpha}, \quad (3.2c)$$

$$[\hat{c}, L_n] = 0, \quad [\hat{c}, J_\alpha] = 0. \quad (3.2d)$$

The *Ramond superalgebra* [501] $\widehat{R}$ is a complex Lie superalgebra with basis $\hat{c}, L_n, n \in \mathbb{Z}, J_\alpha, \alpha \in \mathbb{Z}$, and Lie brackets given again by (3.2).

The Neveu–Schwarz and Ramond superalgebras are also called $N = 1$ *super-Virasoro algebras*, since they can be viewed as $N = 1$ supersymmetry extensions of the Virasoro algebra.

Further, $\widehat{Q}$ will denote $\widehat{W}, \widehat{S}$ or $\widehat{R}$ when a statement holds for all three algebras. The elements $\hat{c}, L_n$ are even and J_α are odd. The grading of $\widehat{Q}$ is given by

$$\deg \hat{c} = 0, \quad \deg L_n = n, \quad \deg J_\alpha = \alpha. \quad (3.3)$$

We have the obvious decomposition

$$\widehat{Q} = \widehat{Q}_+ \oplus \widehat{Q}_0 \oplus \widehat{Q}_- \quad (3.4)$$

where

$$\widehat{Q}_+ = \text{c.l.s.}\{X \in \widehat{Q} \mid \deg X > 0\}, \quad (3.5a)$$

$$\widehat{Q}_0 = \text{c.l.s.}\{X \in \widehat{Q} \mid \deg X = 0\}, \quad (3.5b)$$

$$\widehat{Q}_- = \text{c.l.s.}\{X \in \widehat{Q} \mid \deg X < 0\}. \quad (3.5c)$$

(c.l.s. means ‘complex linear span’.) Thus, $\widehat{W}_0$ and $\widehat{S}_0$ are spanned by $\hat{c}$ and L_0 , while $\widehat{R}_0$ is spanned by $\hat{c}, L_0$ and J_0 . Furthermore, we note that $\widehat{W}_\pm$ is generated by $L_{\pm 1}$ and $L_{\pm 2}$, $\widehat{S}_\pm$ is generated by $J_{\pm \frac{1}{2}}$ and $J_{\pm \frac{3}{2}}$, and $\widehat{R}_\pm$ is generated by $L_{\pm 1}$ and $J_{\pm 1}$.

Next we note that the Cartan subalgebra $\widehat{C}$ of $\widehat{Q}$ is spanned by $\hat{c}$ and L_0 . We note that $\widehat{C} = \widehat{Q}_0$ for $\widehat{W}$ and $\widehat{S}$. In the case of $\widehat{R}$ the generator J_0 is not in $\widehat{C}$, since it is odd and

cannot diagonalize $\widehat{R}_\pm$, having an anti-bracket with the odd generators. Because of this peculiarity we shall use generalized highest weight modules, which will be relevant only in the $\widehat{R}$ case, since in the other two cases they will coincide with ordinary highest weight modules.

A *generalized highest weight module* over $\widehat{Q}$ is characterized by its highest weight $\Lambda \in \widehat{C}^*$ and *generalized highest weight vector* $\tilde{v}_0$, which is a finite-dimensional vector space, so that

$$\begin{aligned} Xv &= 0, \quad X \in \widehat{Q}_+, \quad v \in \tilde{v}_0, \\ Xv &= \Lambda(X)v, \quad X \in \widehat{C}, \quad v \in \tilde{v}_0. \end{aligned} \quad (3.6)$$

We define the generalized highest weight vector $\tilde{v}_0$ for our three algebras by

$$\tilde{v}_0 = \text{c.l.s.}\{v_0\}, \quad \text{for } \widehat{W}, \widehat{S}, \quad (3.7a)$$

$$\tilde{v}_0 = \text{c.l.s.}\{v_0, J_0 v_0\}, \quad \text{for } \widehat{R}, \quad (3.7b)$$

where v_0 fulfils the conditions (3.6), i. e. it is a usual highest weight vector. We denote

$$\Lambda(L_0) = h, \quad \Lambda(\hat{c}) = c. \quad (3.8)$$

Further we shall need also generalized Verma modules over $\widehat{Q}$. Before this we introduce bases in the universal enveloping algebras $U(\widehat{Q}_\pm)$, $U(\widehat{Q}_0)$ as follows:

$$\begin{aligned} J_{\alpha_1} \dots J_{\alpha_k} L_{n_1} \dots L_{n_\ell}, \quad & 0 < \alpha_1 < \dots < \alpha_k, \\ & 0 < n_1 \leq \dots \leq n_\ell, \quad \text{for } \widehat{Q}_+ \end{aligned} \quad (3.9a)$$

$$\begin{aligned} J_{\alpha_1} \dots J_{\alpha_k} L_{n_1} \dots L_{n_\ell}, \quad & \alpha_1 < \dots < \alpha_k < 0, \\ & n_1 \leq \dots \leq n_\ell < 0, \quad \text{for } \widehat{Q}_-, \end{aligned} \quad (3.9b)$$

$$\hat{c}^k L_0^\ell, \quad k, \ell \in \mathbb{Z}_+, \quad \text{for } \widehat{W}_0, \widehat{S}_0, \quad (3.9c)$$

$$\hat{c}^k L_0^\ell J_0^\kappa, \quad k, \ell \in \mathbb{Z}_+, \quad \kappa = 0, 1, \quad \text{for } \widehat{R}_0. \quad (3.9d)$$

Naturally, the odd elements appear at most in first degree, since because of (3.2b) one has

$$J_\alpha^2 = L_{2\alpha}, \quad \alpha \neq 0, \quad (3.10a)$$

$$J_0^2 = L_0 - \frac{1}{16}\hat{c}, \quad \text{for } \widehat{R}_0. \quad (3.10b)$$

For further reference we say that an element u of $U(\widehat{Q})$ is *homogeneous element* if it is an eigenvector of the generator L_0 , i. e. if $[L_0, u] = \lambda_u u$. Then we define the *level of a homogeneous element* u of $U(\widehat{Q})$ as $-\lambda_u$. The basis elements in (3.9a,b) are homogeneous of level $-(\alpha_1 + \dots + \alpha_k + n_1 + \dots + n_\ell)$, (which is negative, positive, respectively, for $U(\widehat{Q}_+)$, $U(\widehat{Q}_-)$, respectively), while those in (3.9c) are homogeneous of level zero.

A generalized Verma module $V^\Lambda = V^{h,c}$ is an induced GHWM with highest weight Λ such that

$$V^\Lambda \cong U(\widehat{Q}) \otimes_{U(\widehat{B})} \tilde{v}_0 \cong \begin{cases} U(\widehat{Q}_-) \otimes \tilde{v}_0 & \text{for } \widehat{W}_0, \widehat{S}_0, \\ U(\widehat{Q}_-) \otimes_{\mathbb{C}J_0} \tilde{v}_0 & \text{for } \widehat{R}_0, \end{cases} \quad (3.11)$$

where $\widehat{B} = \widehat{Q}_+ \oplus \widehat{Q}_0$. Below for simplicity we shall write $U(\widehat{Q}_-) \tilde{v}_0$ omitting the tensor sign. We denote by $L_\Lambda = L^{h,c}$ the irreducible factor module V^Λ / I^Λ , where I^Λ is the maximal proper submodule of V^Λ . Then every irreducible GHWM over $\widehat{Q}$ is isomorphic to some L_Λ .

From now on most formulas will be valid for the three algebras, the different cases being distinguished by two parameters:

$$\mu \equiv \begin{cases} 0 & \text{for } \widehat{W}, \widehat{S}, \\ \frac{1}{2} & \text{for } \widehat{R}, \end{cases} \quad \nu \equiv \begin{cases} 1 & \text{for } \widehat{W}, \\ 2 & \text{for } \widehat{S}, \widehat{R}. \end{cases} \quad (3.12)$$

It is well known (cf. [352] for $\widehat{W}, \widehat{S}$; [245] for $\widehat{R}$) that the generalized Verma module V^Λ is reducible iff h and c are related as follows:

either

$$h = h_{(m,n)} = h_0 + \frac{1}{4}(\alpha_+ m + \alpha_- n)^2 + \frac{1}{8}\mu \quad (3.13a)$$

where,

$$m, n \in \frac{1}{\nu}\mathbb{N}, \quad m - n \in \mathbb{Z} + \mu, \quad (3.13b)$$

$$h_0 = \begin{cases} \frac{1}{24}(c-1) & \text{for } \widehat{W}, \\ \frac{1}{16}(c-1) & \text{for } \widehat{S}, \widehat{R}, \end{cases}$$

$$\alpha_\pm \equiv \begin{cases} \frac{1}{\sqrt{24}}(\sqrt{1-c} \pm \sqrt{25-c}) & \text{for } \widehat{W}, \\ \frac{1}{2}(\sqrt{1-c} \pm \sqrt{9-c}) & \text{for } \widehat{S}, \widehat{R}, \end{cases}$$

or

$$h = \frac{c}{16} \quad \text{for } \widehat{R}. \quad (3.14)$$

For further use we also introduce the notation

$$c_0 \equiv \begin{cases} 25 & \text{for } \widehat{W}, \\ 9 & \text{for } \widehat{S}, \widehat{R}. \end{cases} \quad (3.15)$$

We first comment on the cases from (3.13). We know that the reducible generalized Verma module $V = V^{h,c}$, $h = h_{(m,n)}$, contains a proper submodule isomorphic to the generalized Verma module $V' = V^{h+vmn,c}$. In other words there exists a non-trivial embedding map between V' and V . In this situation we shall use the following pictorial representation:

$$V^{h,c} \rightarrow V^{h+vmn,c}, \quad h = h_{(m,n)}. \quad (3.16)$$

These embedding maps are realized by the so-called singular vectors. A *singular vector* $v_s \in V$ is such that $v_s \neq \tilde{v}_0$ and v_s has the property of the highest weight vector $\tilde{v}'_0$ of V' . Moreover, v_s can be expressed, as an element of $U(\widehat{Q}_-)\tilde{v}_0$, by

$$v_s = \mathcal{P}(\widehat{Q}_-)\tilde{v}_0, \quad V' \cong U(\widehat{Q}_-)\tilde{v}'_0 \cong U(\widehat{Q}_-)\mathcal{P}(\widehat{Q}_-)\tilde{v}_0, \quad (3.17)$$

where $\mathcal{P}(\widehat{Q}_-)$ is a homogeneous polynomial in $U(\widehat{Q}_-)$ of level vmn .

Explicit examples of singular vectors exist for low levels and/or via special constructions, cf. [57, 257, 46, 378, 261] (all for $\widehat{W}$), and also for the cases $h = h_{(m,1)}, h_{(1,n)}$; cf. [60].

Next we would like to understand (3.14). First we note the following decomposition in the $\widehat{R}$ case which holds for all generalized Verma modules:

$$\begin{aligned} V^\Lambda &= V_0^\Lambda \otimes V_1^\Lambda, \\ V_k^\Lambda &= U(\widehat{Q}_-)J_0^k v_0. \end{aligned} \quad (3.18)$$

Let us comment on the action of $\widehat{R}$ on V^Λ . The action of $\widehat{R}_-$ and $\widehat{C}$ preserves each V_k^Λ . Further, we consider separately the cases $h \neq \frac{1}{16}c$ and $h = \frac{1}{16}c$.

1. For $h \neq \frac{1}{16}c$ the action of J_0 is not preserving V_k^Λ . To see this we first note that it mixes the elements of $\tilde{v}_0$,

$$J_0(av_0 + bJ_0v_0) = b\left(h - \frac{1}{16}c\right)v_0 + aJ_0v_0. \quad (3.19)$$

Furthermore, shifting J_0 to the right until it reaches $\tilde{v}_0$ we have to use

$$\begin{aligned} J_0L_n &= L_nJ_0 - \frac{1}{2}nJ_n, \quad n < 0, \\ J_0J_\alpha &= -J_\alphaJ_0 + 2L_\alpha, \quad \alpha < 0. \end{aligned} \quad (3.20)$$

Finally, the action of $\widehat{R}_+$ is not preserving V_k^Λ , since shifting its elements to the right one produces also J_0 , since

$$\begin{aligned} L_nJ_{-n} &= J_{-n}L_n + \frac{3}{2}nJ_0, \quad n > 0, \\ J_nL_{-n} &= L_{-n}J_n + \frac{3}{2}nJ_0, \quad n > 0. \end{aligned} \quad (3.21)$$

2. In the case $h = \frac{1}{16}c$ the subspace V_1^Λ is preserved by the action of $\widehat{R}$. To see this it is enough to note that $J_0(J_0v_0) = (h - \frac{1}{16}c)v_0 = 0$. Thus, the action of J_0 and, consecutively, of $\widehat{R}_+$ on V_1^Λ cannot produce elements of V_0^Λ .

Thus, for $h = \frac{1}{16}c$ the generalized Verma module $V = V^{c/16,c}$ over $\widehat{R}$ is reducible. It contains a proper submodule $V_1^{c/16,c}$ which is isomorphic to an ordinary Verma module $\tilde{V} = \tilde{V}^{c/16,c}$ with the same highest weight and highest weight vector $v'_0 = J_0v_0$ with the

additional property $J_0 v'_0 = 0$. The factor module $V^{c/16,c}/V_1^{c/16,c}$ is also isomorphic to $\tilde{V}$ since it has the same highest weight and its highest weight vector $v''_0 = v_0$ also fulfils the property to be annihilated by J_0 ; indeed, we have $J_0 v''_0 = 0$, since $J_0 v_0 \in V_1^{c/16,c}$. For this reason, furthermore, in the case $h = \frac{1}{16}c$ we shall consider $\tilde{V}$ instead of $V^{c/16,c}$ for $\hat{R}$.

It is also possible that (3.13) and (3.14) hold simultaneously. In this case $\tilde{V}$ is further reducible and everything we said for cases from (3.13) applies also to this case. Furthermore, combining (3.13) and (3.14) we have

$$h = \frac{1}{16}c = h_{(m,n)} = \frac{1}{16}(c-1) + \frac{1}{4}(m\alpha_+ + n\alpha_-)^2 + \frac{1}{16} \quad (3.22a)$$

$$\Rightarrow m\alpha_+ + n\alpha_- = 0 \quad (3.22b)$$

$$\Rightarrow -\frac{\alpha_-}{\alpha_+} = \frac{m}{n} = \frac{2m}{2n} = \frac{p}{q}. \quad (3.22c)$$

Combining this with the requirement that $m-n \in \mathbb{Z} + \frac{1}{2}$ or $2m-2n \in 2\mathbb{Z} + 1$, we see that one of the two numbers $2m, 2n$ must be odd and the other even, and then one of the two numbers p, q must be odd and the other even. Thus, we see that the only possibility in this case is

$$m = \frac{1}{2}p, \quad n = \frac{1}{2}q, \quad pq \in 2\mathbb{N}. \quad (3.23)$$

3.2 Unitarity

The irreducible factor modules $L_\Lambda = L^{h,c} = V^\Lambda/I^\Lambda$ are unitarizable [245, 276, 406] if either $h \geq 0$, $c \geq 1$, or when

$$h = \frac{1}{4vp(p+v)}[v^2(pn-m(p+v))^2 - v^2] + \frac{1}{8}\mu, \quad c = 1 - \frac{2(2+v)}{p(p+v)}, \quad p = 2, 3, \dots, \quad (3.24)$$

or when

$$h = h_{(\frac{p}{2}, \frac{p+1}{2})}^- = \frac{1}{16}c_{p,p+1}^- = \frac{1}{16}\left(1 - \frac{2}{p(p+1)}\right), \quad p = 1, 2, \dots, \text{ for } \hat{R}. \quad (3.25)$$

The cases (3.24) are the modules $L_{00} \equiv L_{10}$ (parametrized by (m, n) as in Table 3.2) which form the $c < 1$ series of unitarizable HWM over $\widehat{W}$, [245, 406], $\hat{S}$, $\hat{R}$ [245, 276] inside the so-called Kac table [57]. The case (3.25) is the module $L_{00}^0 \equiv L_{10}^0$ on the border of the Kac table.

In the cases (3.24) for $p = 2$ one gets the trivial case $c = 0$. The cases $p = 3, 4, 5, 6$ for the Virasoro algebra ($v = 1$), i. e. $c = 1/2, 7/10, 4/5, 5/6$, respectively, correspond to the Ising model, the tri-critical Ising model, the 3-state Potts model, and the tri-critical 3-state Potts model, respectively [245]. Note that there is only one value of the central charge which is common for the Virasoro and super-Virasoro algebras. This is the case $c = 7/10$ for Virasoro ($p = 4$) and $c = 7/15$ for the super-Virasoro algebras ($p = 3$), taking into account that $c_{\widehat{W}} = \frac{3}{2}c_{\hat{S}, \hat{R}}$ (compare (3.1a) and (3.2a)).

3.3 Multiplet classification of the reducible (generalized) Verma modules

We used extensively the notion of multiplet in the previous volumes. Now we recall the definition of a multiplet following [153] (and Volumes 1 and 2):

Definition. A set $\mathcal{M}$ of HWMs is said to form a multiplet if: 1. For each $M \in \mathcal{M}$ there exists another $M' \in \mathcal{M}$ which is partially equivalent to M . 2. For each $M \in \mathcal{M}$ all HWMs which are partially equivalent to it belong to $\mathcal{M}$. 3. $\mathcal{M}$ does not have proper subsets which fulfil 1. and 2. $\diamond$

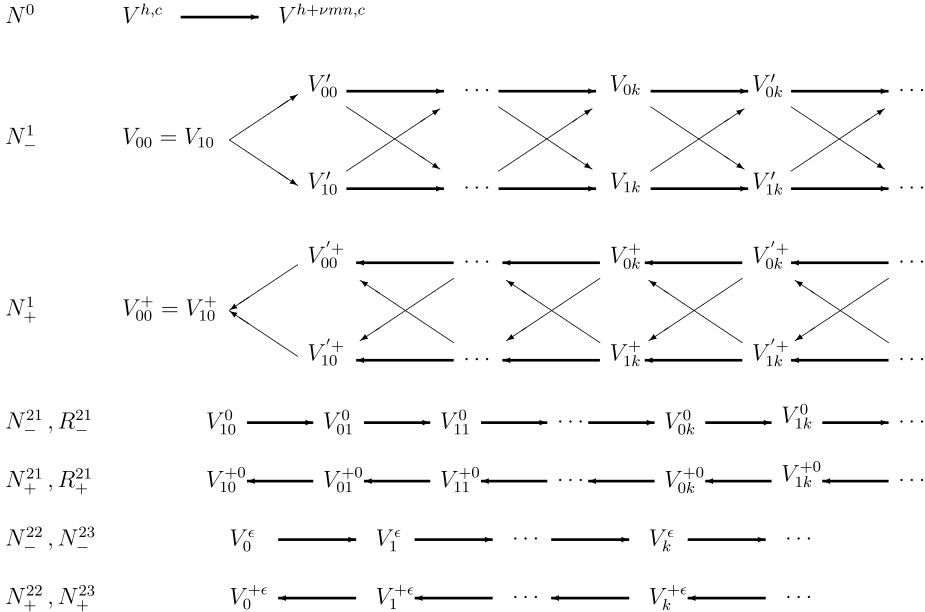
We shall use pictorial representations of multiplets as connected graphs with vertices corresponding to the HWM and links corresponding to the intertwining maps between partially equivalent HWM.

Furthermore, we restrict our multiplets to the category Verma modules.

Here we present the multiplet classification of the reducible (generalized) Verma modules over $\widehat{W}$ [214] (following [155]) and over $\widehat{S}, \widehat{R}$ [156]. There are five types in each case which will be denoted (following [156]) $N^0, N_+^1, N_-^1, N_+^2, N_-^2$. They are shown in Table 3.1. (Note that in [215, 155] the notation was II, $\text{III}_\pm, \text{III}_\pm^0, (\text{III}_\pm^{00})$, respectively.)

The type N^0 occurs when the ratio α_-/α_+ is not a real rational number.

Table 3.1: Types of multiplets (embedding diagrams) of reducible Verma modules over the Virasoro and $N = 1$ super-Virasoro algebras (arrows point to the embedded modules).



For all other types the ratio α_-/α_+ is a real rational number and *either* $c \leq 1$ or $c \geq c_0$.

For $c < 1$ we have type N_-^1 and subtypes N_-^{21}, N_-^{22} of type N_-^2 . In this case the ratio α_-/α_+ is negative so we set $\alpha_-/\alpha_+ = -p/q, p, q \in \mathbb{N}, p \nmid q$ (the latter means that p, q have no common divisors), and then we have

$$\begin{aligned} c = c_{p,q}^- &= \frac{1}{2}(c_0 + 1) - \frac{1}{4}(c_0 - 1)\left(\frac{p}{q} + \frac{q}{p}\right) = 1 - (10 - 4v)\frac{(p - q)^2}{pq} \\ &= \begin{cases} 13 - 6(\frac{p}{q} + \frac{q}{p}) & \text{for } \widehat{W} \\ 5 - 2(\frac{p}{q} + \frac{q}{p}) & \text{for } \widehat{S}, \widehat{R} \end{cases} \quad (c < 1). \end{aligned} \quad (3.26)$$

Substituting $c \rightarrow c_{p,q}^-$ in (3.13a) we get

$$h = h_{(m,n)}^- = \frac{1}{4vpq} [v^2(pn - qm)^2 - (p - q)^2] + \frac{1}{8}\mu. \quad (3.27)$$

For $c = 1$ we have subtype N_-^{23} of type N_-^2 .

For $c > c_0$ we have type N_+^1 and subtypes N_+^{21}, N_+^{22} of type N_+^2 . Here the ratio α_-/α_+ is positive so we set $\alpha_-/\alpha_+ = p/q, p, q \in \mathbb{N}, p \nmid q$, and then we have

$$\begin{aligned} c = c_{p,q}^+ &= \frac{1}{2}(c_0 + 1) + \frac{1}{4}(c_0 - 1)\left(\frac{p}{q} + \frac{q}{p}\right) = 1 + (10 - 4v)\frac{(p - q)^2}{pq} \\ &= \begin{cases} 13 + 6(\frac{p}{q} + \frac{q}{p}) & \text{for } \widehat{W} \\ 5 + 2(\frac{p}{q} + \frac{q}{p}) & \text{for } \widehat{S}, \widehat{R} \end{cases} \quad (c > c_0). \end{aligned} \quad (3.28)$$

Substituting $c \rightarrow c_{p,q}^+$ in (3.13a) we get

$$h = h_{(m,n)}^+ = \frac{1}{4vpq} [(p + q)^2 - v^2(pn - qm)^2] + \frac{1}{8}\mu = \frac{1}{v} + \frac{1}{4}\mu - h_{(m,n)}^-. \quad (3.29)$$

For $c = c_0$ we have subtype N_+^{23} of type N_+^2 .

The explicit parametrization of all types and subtypes is given in Table 3.2 (for $\widehat{W}$ see [215] and Propositions 1.1–1.5, formulae (6), (13), (16) of [155], for $\widehat{S}, \widehat{R}$ see Propositions 1–4, formulae (6), (11), and (14) of [156]). Note that in each case the (sub)types N_- with $c \leq 1$ have the same parametrization as the corresponding N_+ with $c \rightarrow c_0 + 1 - c \geq c_0$. Further, we note that the parametrization of subtypes $N_{\pm}^{21}$ is obtained from the parametrization of $N_{\pm}^1$ by formally setting $n = \tilde{q}$ and replacing the condition $p < q$ by $p \neq q$. Next, the parametrization of subtypes $N_{\pm}^{22}$ is obtained from the parametrization of $N_{\pm}^{21}$ by formally setting $m = \tilde{p}$ or $m = 0$. Finally, the parametrization of subtypes $N_{\pm}^{23}$ is obtained from the parametrization of $N_{\pm}^{22}$ by formally setting $p = q$ (which forces $p = q = 1$).

Further we give the explicit parametrization of the (generalized) Verma modules in the different multiplets. The cases N^0 are simple and all info about them is already present in Tables 3.1 and 3.2.

Table 3.2: Parametrization of the types of multiplets of (generalized) Verma modules over the Virasoro and $N = 1$ super-Virasoro algebras.

type	parameters	range and constraints on parameters	remarks
N^0 (II)	c, m, n	$c \in \mathbb{C}, \alpha_-/\alpha_+ \notin \mathbb{Q},$ $m, n \in \frac{1}{\nu}\mathbb{N}, m - n \in \mathbb{Z} + \mu$	
$N_{\pm}^1$ (III $_{\pm}$)	p, q, m, n	$p, q \in \mathbb{N}, pXq, p < q,$ $m, n \in \frac{1}{l}\mathbb{N}, m - n \in \mathbb{Z} + \mu,$ $pn > qm, n < \tilde{q},$ $\tilde{q} \equiv \begin{cases} \frac{q}{2} & \text{if } pq \in 2\mathbb{N} + 1 \text{ for } \hat{S}, \hat{R} \\ q & \text{otherwise} \end{cases}$	$\alpha_-/\alpha_+ = \pm p/q; c = c_{p,q}^{\pm};$
$N_{\pm}^{21}$ (III $_{\pm}^0$)	p, q, m	$p, q \in \mathbb{N}, pXq, p \neq q, m \in \frac{1}{l}\mathbb{N},$ $m < \tilde{p} \neq \tilde{q}, m - \tilde{q} \in \mathbb{Z} + \mu$	$\alpha_-/\alpha_+ = \pm p/q \text{ or } \pm q/p;$ $c = c_{pq}^{\pm};$
$R_{\pm}^{21}$	p, q	$p, q \in \mathbb{N}, pXq, p \neq q, pq \in 2\mathbb{N},$ $h = h_{(\frac{p}{2}, \frac{q}{2})}^{\pm} = \frac{1}{16} c_{p,q}^{\pm}, \text{ only for } \hat{R}$	$\alpha_-/\alpha_+ = \pm p/q, c = c_{pq}^{\pm};$ the only possibility for $\hat{V}_{\frac{c}{16}, c}$
$N_{\pm}^{22}$ (III $_{\pm}^0$)	p, q, ϵ	$p, q \in \mathbb{N}, pXq, p < q,$ $\epsilon = 0, 1, \tilde{q} - \epsilon\tilde{p} \in \mathbb{Z} + \mu$	$\alpha_-/\alpha_+ = \pm p/q; c = c_{pq}^{\pm}$
$N_{\pm}^{23}$ (III $_{\pm}^{00}$)	ϵ	$\epsilon = 0, 1 \text{ for } \hat{W},$ $\epsilon = 0 \text{ for } \hat{S}, \epsilon = 1 \text{ for } \hat{R},$	$c = c_{11}^{-} = 1 \text{ for } N_{-}^{23},$ $c = c_{11}^{+} = c_0 \text{ for } N_{+}^{23}$

We start with type N_{-}^1 . For fixed parameters p, q, m, n (cf. Table 3.2) the (generalized) Verma modules of this type form a multiplet represented by the corresponding commutative diagram of Table 3.1. The modules of this multiplet are divided in four infinite groups which are given as follows (cf. [155], formula (10), [156], formula (10)):

$$V_{0k} = V^{h_{0k}, c}, \quad k \in \mathbb{Z}_{+},$$

$$h_{0k} = h_{00} + \nu k(\tilde{p}\tilde{q}k + \tilde{q}m - \tilde{p}n) \quad (3.30a)$$

$$= h_{(2k\tilde{p}+m, n)}^{-} = h_{(\tilde{p}-m, 2k\tilde{q}+\tilde{q}-n)}^{-};$$

$$V_{1k} = V^{h_{1k}, c}, \quad k \in \mathbb{Z}_{+},$$

$$h_{1k} = h_{00} + \nu k(\tilde{p}\tilde{q}k - \tilde{q}m + \tilde{p}n) \quad (3.30b)$$

$$= h_{(m, 2k\tilde{q}+n)}^{-} = h_{(2k\tilde{p}+\tilde{p}-m, \tilde{q}-n)}^{-};$$

$$V'_{0k} = V^{h'_{0k}, c}, \quad k \in \mathbb{Z}_{+},$$

$$h'_{0k} = h_{00} + \nu(\tilde{q}k + n)(\tilde{p}k + m) \quad (3.30c)$$

$$= h_{(2k\tilde{p}+\tilde{p}+m, \tilde{q}-n)}^{-} = h_{(\tilde{p}-m, 2k\tilde{q}+\tilde{q}+n)}^{-};$$

$$V'_{1k} = V^{h'_{1k}, c}, \quad k \in \mathbb{Z}_{+},$$

$$h'_{1k} = h_{00} + \nu(\tilde{q}k + \tilde{q} - n)(\tilde{p}k + \tilde{p} - m) \quad (3.30d)$$

$$= h_{(m, 2k\tilde{q}+2\tilde{q}-n)}^{-} = h_{(2k\tilde{p}+2\tilde{p}-m, n)}^{-},$$

where $c = c_{p,q}^-$, and we note that $h_{00} = h_{10}$. It is easy to understand the embedding structure having in mind the basic fact about the embeddings represented by (3.16) and the explicit representation of the conformal weights in (3.30).

Let us start with the Verma module $V_{00} = V_{10}$ which is on the tip of the embedding diagram. We note the following (periodicity) properties for its weight:

$$h_{(m,n)}^- = h_{(\tilde{p}-m, \tilde{q}-n)}^- \quad (3.31a)$$

$$= h_{(m+\ell\tilde{p}, n+\ell\tilde{q})}^- = h_{(\tilde{p}-m+\ell\tilde{p}, \tilde{q}-n+\ell\tilde{q})}^-, \quad \ell \in \mathbb{Z}_+. \quad (3.31b)$$

The above means that the Verma module $V^{h_{(m,n)}^-, c}$ has an infinite number of Verma submodules, namely, using the rule (3.16) we have

$$\begin{aligned} V^{h_{(m,n)}^-, c} &\rightarrow V^{h_{(m,n)}^- + v(m+\ell\tilde{p})(n+\ell\tilde{q}), c} \\ V^{h_{(m,n)}^-, c} &\rightarrow V^{h_{(m,n)}^- + v(\tilde{p}-m+\ell\tilde{p})(\tilde{q}-n+\ell\tilde{q}), c}. \end{aligned} \quad (3.32)$$

However, only the lowest level embeddings in each sequence of (3.32), i. e. the cases $\ell = 0$, are elementary embeddings. These are the embeddings of V'_{00} , V'_{10} , respectively. All other embeddings are compositions of elementary embeddings.

Similar facts hold for all other modules in the multiplet. Namely, each module has an infinite number of Verma submodules, and again only two embeddings are elementary. We give now all such elementary embeddings explicitly together with the weight differences, noting that the analogue of property (3.31a) (which is essential for the elementary embeddings) is given for each Verma module in formulae (3.30):

$$V_{0k} \rightarrow V'_{0k}, \quad h'_{0k} - h_{0k} = v n(2k\tilde{p} + m), \quad (3.33a)$$

$$V_{0k} \rightarrow V'_{1k}, \quad h'_{1k} - h_{0k} = v(\tilde{p} - m)(2k\tilde{q} + \tilde{q} - n), \quad (3.33b)$$

$$V_{1k} \rightarrow V'_{0k}, \quad h'_{0k} - h_{1k} = v m(2k\tilde{q} + n), \quad (3.33c)$$

$$V_{1k} \rightarrow V'_{1k}, \quad h'_{1k} - h_{1k} = v(\tilde{q} - n)(2k\tilde{p} + \tilde{p} - m), \quad (3.33d)$$

$$V'_{0k} \rightarrow V_{0,k+1}, \quad h_{0,k+1} - h'_{0k} = v(\tilde{q} - n)(2k\tilde{p} + \tilde{p} + m), \quad (3.33e)$$

$$V'_{0k} \rightarrow V_{1,k+1}, \quad h_{1,k+1} - h'_{0k} = v(\tilde{p} - m)(2k\tilde{q} + \tilde{q} + n), \quad (3.33f)$$

$$V'_{1k} \rightarrow V_{0,k+1}, \quad h_{0,k+1} - h'_{1k} = v m(2k\tilde{q} + 2\tilde{q} - n), \quad (3.33g)$$

$$V'_{1k} \rightarrow V_{1,k+1}, \quad h_{1,k+1} - h'_{1k} = v n(2k\tilde{p} + 2\tilde{p} - m). \quad (3.33h)$$

Of course, the weight differences in (3.33) give the levels of the singular vectors generating these embeddings.

The (generalized) Verma modules of type N_+^1 for fixed parameters p, q, m, n also form a multiplet represented by the corresponding commutative diagram of Table 3.1. The modules of this multiplet are also divided in four infinite groups which we denote by $V_{0k}^+, V_{0k}'^+, V_{1k}^+, V_{1k}'^+$, which are given by (3.30) with the changes $\tilde{p} \rightarrow -\tilde{p}$, $m \rightarrow -m$ (or $\tilde{q} \rightarrow -\tilde{q}$, $n \rightarrow -n$), $h \rightarrow h_{(m,n)}^+$, $c \rightarrow c_{p,q}^+$. From (3.33) using these changes we obtain also

the weight differences of the elementary embeddings which are now in the opposite direction.

We continue with subtype N_-^{21} . For fixed parameters p, q, m (cf. Table 3.2) the (generalized) Verma modules of this type form a multiplet represented by the corresponding diagram of Table 3.1. The modules of this multiplet are divided in two infinite groups whose parametrization is obtained from those of type N_-^1 by setting $n = \tilde{q}$,

$$\begin{aligned} V_{0k}^0 &= (V_{0k})|_{n=\tilde{q}} = (V'_{0,k-1})|_{n=\tilde{q}, k \geq 1} = V_{0k}^{h_{0k}^0, c}, \quad k \in \mathbb{Z}_+, \\ h_{0k}^0 &= h_{00}^0 + vk\tilde{q}(\tilde{p}k + m - \tilde{p}) = h_{(\tilde{p}-m, 2k\tilde{q})}^-, \end{aligned} \quad (3.34a)$$

$$\begin{aligned} V_{1k}^0 &= (V_{1k})|_{n=\tilde{q}} = (V'_{1k})|_{n=\tilde{q}} = V_{1k}^{h_{1k}^0, c}, \quad k \in \mathbb{Z}_+, \\ h_{1k}^0 &= h_{10}^0 + vk\tilde{q}(\tilde{p}k - m + \tilde{p}) = h_{(m, (2k+1)\tilde{q})}^-, \end{aligned} \quad (3.34b)$$

where $h_{0k}^0 = h_{1k}^0 = h_{(m, \tilde{q})}^- = \frac{1}{4vpq}[v^2q^2(\tilde{p} - m)^2 - (p - q)^2] + \frac{1}{8}\mu$ and $c = c_{p,q}^-$. Note that $V_{00}^0 = V_{10}^0$. It is easy to understand the embedding structure using formulae (3.33) with $n = \tilde{q}$. Thus we see that embeddings (3.33a,h) become the composite embeddings $V_{0k}^0 \rightarrow V_{0,k+1}^0$, $V_{1k}^0 \rightarrow V_{1,k+1}^0$, respectively, embeddings (3.33b,f) coincide (up to shift in k), embeddings (3.33c,g) coincide, embeddings (3.33d,e) vanish. Explicitly, the elementary embeddings are

$$V_{0k}^0 \rightarrow V_{1k}^0, \quad h_{1k}^0 - h_{0k}^0 = 2vk\tilde{q}(\tilde{p} - m), \quad k \in \mathbb{N}, \quad (3.35a)$$

$$V_{1k}^1 \rightarrow V_{0,k+1}^0, \quad h_{0,k+1}^0 - h_{1k}^0 = vm\tilde{q}(2k + 1), \quad k \in \mathbb{Z}_+. \quad (3.35b)$$

The multiplet of subtype N_{21}^- describes also the situation for $\hat{R}$ when both (3.13) and (3.14) hold, and then (3.23) holds. We shall denote the corresponding multiplet by R_{21}^- to stress that it happens only for $\hat{R}$. The modules of this multiplet are also divided in two infinite groups whose parametrization is obtained from those of type N_-^1 by using (3.23). Thus, we get

$$\begin{aligned} \tilde{V}_{0k}^0 &= (\tilde{V}_{0k})|_{(m,n)=(\frac{p}{2}, \frac{q}{2})} = (\tilde{V}_{1k})|_{(m,n)=(\frac{p}{2}, \frac{q}{2})} = \tilde{V}_{0k}^{\tilde{h}_{0k}^0, c}, \quad k \in \mathbb{Z}_+, \\ \tilde{h}_{0k}^0 &= \tilde{h}_{00}^0 + 2pqk^2 = h_{((4k+1)\frac{p}{2}, \frac{q}{2})}^- = h_{(\frac{p}{2}, (4k+1)\frac{q}{2})}^-, \end{aligned} \quad (3.36a)$$

$$\begin{aligned} \tilde{V}_{1k}^0 &= (\tilde{V}'_{0k})|_{(m,n)=(\frac{p}{2}, \frac{q}{2})} = (\tilde{V}'_{1k})|_{(m,n)=(\frac{p}{2}, \frac{q}{2})} = \tilde{V}_{1k}^{\tilde{h}_{1k}^0, c}, \quad k \in \mathbb{Z}_+, \\ \tilde{h}_{1k}^0 &= \tilde{h}_{10}^0 + 2pq\left(k + \frac{1}{2}\right)^2 = h_{((4k+3)\frac{p}{2}, \frac{q}{2})}^- = h_{(\frac{p}{2}, (4k+3)\frac{q}{2})}^-, \end{aligned} \quad (3.36b)$$

where $\tilde{h}_{00}^0 = h_{(\frac{p}{2}, \frac{q}{2})}^- = -\frac{1}{8pq}(p - q)^2 + \frac{1}{16} = \frac{c}{16} = \frac{1}{16}c_{p,q}^-$. We recall that the modules here are ordinary Verma modules with highest weight vector J_0v_0 . Note that the tildes are omitted in Table 3.1, since we use the same diagram as for N_{21}^- . It is easy to understand the embedding structure using formulae (3.33) with $(m, n) = (\frac{p}{2}, \frac{q}{2})$. Thus we see

that embeddings (3.33a,b,c,d) coincide, and embeddings (3.33e,f,g,h) also coincide. Explicitly, the elementary embeddings are

$$\widetilde{V}_{0k}^0 \rightarrow \widetilde{V}_{1k}^0, \quad \tilde{h}_{1k}^0 - \tilde{h}_{0k}^0 = \frac{1}{2}pq(4k+1), \quad (3.37a)$$

$$\widetilde{V}_{1k}^0 \rightarrow \widetilde{V}_{0,k+1}^0, \quad \tilde{h}_{0,k+1}^0 - \tilde{h}_{1k}^0 = \frac{1}{2}pq(4k+3). \quad (3.37b)$$

The (generalized) Verma modules of subtype N_+^{21} for fixed parameters p, q, m also form a multiplet represented by the corresponding diagram of Table 3.1. The modules of this multiplet are also divided in two groups which we denote by V_{0k}^{+0}, V_{1k}^{+0} , which are given by (3.34) with the changes $\tilde{p} \rightarrow -\tilde{p}, m \rightarrow -m, h \rightarrow h_{(m,\tilde{q})}^+ = \frac{1}{4vpq}[(p+q)^2 - v^2q^2(\tilde{p}-m)^2] + \frac{1}{8}\mu, c \rightarrow c_{p,q}^+$. From (3.35) using these changes we obtain also the weight differences of the elementary embeddings which are now in the opposite direction.

The multiplet N_{21}^+ describes also the situation for $\hat{R}$ when both (3.13) and (3.14) hold, and then (3.23) holds. The corresponding multiplet is denoted by R_{21}^+ to stress that it happens only for $\hat{R}$. The modules of this multiplet are also divided in two infinite groups whose parametrization is obtained from those of type N_+^1 by using (3.23). Thus, we get

$$\begin{aligned} \widetilde{V}_{0k}^{+0} &= (\widetilde{V}_{0k}^+)|_{(m,n)=(\frac{p}{2}, \frac{q}{2})} = (\widetilde{V}_{1k}^+)|_{(m,n)=(\frac{p}{2}, \frac{q}{2})} = \widetilde{V}_{0k}^{\tilde{h}_{0k}^{+0}, c}, \\ \tilde{h}_{0k}^{+0} &= \tilde{h}_{00}^{+0} - 2pqk^2, \end{aligned} \quad (3.38a)$$

$$\begin{aligned} \widetilde{V}_{1k}^{+0} &= (\widetilde{V}_{0k}^{' +})|_{(m,n)=(\frac{p}{2}, \frac{q}{2})} = (\widetilde{V}_{1k}^{' +})|_{(m,n)=(\frac{p}{2}, \frac{q}{2})} = \widetilde{V}_{1k}^{\tilde{h}_{1k}^{+0}, c}, \\ \tilde{h}_{1k}^{+0} &= \tilde{h}_{10}^{+0} - 2pq\left(k + \frac{1}{2}\right)^2, \end{aligned} \quad (3.38b)$$

where $k \in \mathbb{Z}_+, \tilde{h}_{00}^{+0} = h_{(\frac{p}{2}, \frac{q}{2})}^+ = \frac{1}{8pq}(p+q)^2 + \frac{1}{16} = \frac{c}{16} = \frac{1}{16}c_{p,q}^+$. We recall that the modules here are ordinary Verma modules with highest weight vector J_0v_0 . Note that the tildes are omitted in Table 3.1 since we use the same diagram as for N_{21}^+ . From (3.37) using the formal change $p \rightarrow -p$ (or $q \rightarrow -q$) we obtain also the weight differences of the elementary embeddings which are now in the opposite direction.

We continue with subtype N_-^{22} . For fixed parameters p, q, ϵ (cf. Table 3.2) the (generalized) Verma modules of this subtype form a multiplet represented by the corresponding diagram of Table 3.1. The modules of the multiplet are an infinite set whose parametrization is obtained from those of subtype N_-^{21} by setting $m = \tilde{p}$ when $\epsilon = 0$,

$$\begin{aligned} V_k^0 &= (V_{0k}^0)|_{m=\tilde{p}} = (V_{1k}^0)|_{m=\tilde{p}} = V_k^{h_k^0, c}, \\ h_k^0 &= h_0^0 + v\tilde{q}\tilde{p}k^2 = h_{(\tilde{p}, (2k+1)\tilde{q})}^-, \end{aligned} \quad (3.39a)$$

where $k \in \mathbb{Z}_+, h_0^0 = h_{(\tilde{p}, \tilde{q})}^- = -\frac{1}{4vpq}(p-q)^2 + \frac{1}{8}\mu$ and $c = c_{p,q}^-$, or setting $m = 0$ when $\epsilon = 1$

$$\begin{aligned} V_k^1 &= (V_{1k}^0)|_{m=0} = (V_{0,k+1}^0)|_{m=0} = V_k^{h_k^1, c}, \\ h_k^1 &= h_0^1 + v\tilde{p}\tilde{q}k(k+1) = h_{(\tilde{p}, 2(k+1)\tilde{q})}^-, \end{aligned} \quad (3.39b)$$

where $k \in \mathbb{Z}_+$, $h_0^1 = h_{(0,\tilde{q})}^- = \frac{1}{4vpq} [v^2 p^2 \tilde{q}^2 - (p - q)^2] + \frac{1}{8}\mu$ and $c = c_{p,q}^-$. Note that the case (3.39a) it is necessary that $\tilde{p} - \tilde{q} \in \mathbb{Z} + \mu$, which excludes the $\widehat{R}$ case since $\tilde{p} - \tilde{q} \in \mathbb{Z}$ in all cases. In the case (3.39b) it is necessary that $\tilde{p} \in \mathbb{Z} + \mu$, which excludes the $\widehat{R}$ case if $pq \in 2\mathbb{N}$ and excludes the $\widehat{S}$ case if $pq \in 2\mathbb{N} + 1$. It is easy to understand the embedding structure using formulae (3.35) with $m = \tilde{p}$, $m = 0$, respectively. Thus we see that embedding (3.35a), (3.35b), respectively, becomes trivial, and the nontrivial are the ones obtained from (3.35b), (3.35a), respectively. Explicitly, the elementary embeddings are

$$V_k^0 \rightarrow V_{k+1}^0, \quad h_{k+1}^0 - h_k^0 = v\tilde{p}\tilde{q}(2k+1), \quad k \in \mathbb{Z}_+, \quad (3.40a)$$

$$V_k^1 \rightarrow V_{k+1}^1, \quad h_{k+1}^1 - h_k^1 = 2v\tilde{p}\tilde{q}(k+1), \quad k \in \mathbb{Z}_+. \quad (3.40b)$$

The (generalized) Verma modules of subtype N_+^{22} for fixed parameters p, q, ϵ also form a multiplet represented by the corresponding diagram of Table 3.1. The modules of this multiplet also form an infinite set we denote by $V_k^{+\epsilon}$, whose parametrization is obtained from (3.39a) and (3.39b), respectively, for $\epsilon = 0, 1$, respectively, with the changes $h \rightarrow h_{(\tilde{p},\tilde{q})}^+ = \frac{1}{4vpq} (p+q)^2 + \frac{1}{8}\mu$, $h \rightarrow h_{(0,\tilde{q})}^+ = \frac{1}{4vpq} [(p+q)^2 - v^2 q^2 \tilde{p}^2] + \frac{1}{8}\mu$, respectively, and $\tilde{p} \rightarrow -\tilde{p}$, $c \rightarrow c_{p,q}^+$. From (3.40) using the formal change $p \rightarrow -p$ (or $q \rightarrow -q$) we obtain also the weight differences of the elementary embeddings which are now in the opposite direction.

We continue with subtype N_-^{23} . For fixed parameter $\epsilon = 0, 1$ (cf. Table 3.2) the (generalized) Verma modules of this subtype form a multiplet represented by the corresponding diagram of Table 3.1. The (generalized) Verma modules of this subtype form an infinite set whose parametrization is obtained from those of subtype N_-^{22} by setting $p = q = 1$; then $\tilde{p} = \tilde{q} = \frac{1}{v}$; $c = c_{1,1}^- = 1$. We have for $\epsilon = 0$

$$V_k^0 = V^{h+\frac{1}{v}k^2,1} \quad (3.41)$$

where $k \in \mathbb{Z}_+$, $h = h_{(\frac{1}{2},\frac{1}{2})}^- = \frac{1}{8}\mu = 0$. The last equality holds because this case is possible only for $\widehat{W}, \widehat{S}$. For $\epsilon = 1$ we have

$$V_k^1 = V^{h+\frac{1}{v}k(k+1),1} \quad (3.42)$$

where $k \in \mathbb{Z}_+$, $h = h_{(0,\frac{1}{2})}^- = \frac{1}{4v} + \frac{1}{8}\mu$. This case is possible only for $\widehat{W}, \widehat{R}$. From (3.40) we obtain also the weight differences of the elementary embeddings by setting $\tilde{p} = \tilde{q} = \frac{1}{v}$.

Finally, we consider subtype N_+^{23} . For fixed parameter $\epsilon = 0, 1$ (cf. Table 3.2) the (generalized) Verma modules of this subtype form a multiplet represented by the corresponding diagram of Table 3.1. The (generalized) Verma modules of this subtype form an infinite set whose parametrization is obtained from those of subtype N_+^{22} by setting $p = q = 1$; then $\tilde{p} = \tilde{q} = \frac{1}{v}$; $c = c_{1,1}^+ = c_0$. We have for $\epsilon = 0$

$$V_k^{+0} = V^{h-\frac{1}{v}k^2,1} \quad (3.43)$$

where $k \in \mathbb{Z}_+$, $h = h_{(\frac{1}{2}, \frac{1}{2})}^+ = \frac{1}{v} + \frac{1}{8}\mu = \frac{1}{v}$. The last equality is because this case is possible only for $\widehat{W}, \widehat{S}$. For $\epsilon = 1$ we have

$$V_k^1 = V^{h - \frac{1}{v}k(k+1), 1} \quad (3.44)$$

where $k \in \mathbb{Z}_+$, $h = h_{(0, \frac{1}{2})}^+ = \frac{3}{4v} + \frac{1}{8}\mu$. This case is possible only for $\widehat{W}, \widehat{R}$. From (3.40) using the formal change $p \rightarrow -p$ (or $q \rightarrow -q$) and setting $\tilde{p} = \tilde{q} = \frac{1}{v}$ we obtain also the weight differences of the elementary embeddings which are now in the opposite direction.

3.4 Characters of (generalized) highest weight modules

We recall the weight space decomposition of $V^{h,c}$

$$V^{h,c} = \bigoplus_j V_j^{h,c}, \quad j \in \mathbb{Z}_+ \text{ for } \widehat{W}, \widehat{R}, j \in \frac{1}{2}\mathbb{Z}_+ \text{ for } \widehat{S}, \quad (3.45)$$

where $V_j^{h,c}$ are eigenspaces of L_0

$$V_j^{h,c} = \{v \in V^{h,c} \mid L_0 v = (h + j)v\} \cong U(\widehat{Q}_-) \tilde{v}_0, \quad (3.46)$$

where the last equality follows from

$$U(\widehat{Q}_-) = \bigoplus_j U(\widehat{Q}_-)_j \quad (3.47)$$

with the range of j as in (16). For $\widehat{R}$ we have also

$$\begin{aligned} V_j^{h,c} &= V_{j,1}^{h,c} \oplus V_{j,2}^{h,c}, \\ V_{j,1}^{h,c} &= U(\widehat{Q}_-) j_0 v_0, \quad V_{j,2}^{h,c} = U(\widehat{Q}_-) j v_0. \end{aligned} \quad (3.48)$$

The character of $V^{h,c}$ is defined (cf. [354, 245]) as

$$\text{ch } V^{h,c}(t) = \sum_j (\dim V_j^{h,c}) t^{h+j} = t^h \sum_j p(j) t^j = t^h \psi(t), \quad (3.49)$$

where $p(j)$ is the partition function ($p(j) = \#$ of ways j can be represented as the sum of positive integers (and half-integers for $\widehat{S}$); $p(0) \equiv 1$), while $\psi(t)$ is given by [354, 245]:

$$\psi(t) = \begin{cases} \prod_{k \in \mathbb{N}} (1 - t^k)^{-1} & \text{for } \widehat{W}, \\ \prod_{k \in \mathbb{N}} (1 + t^{k-1/2}) / (1 - t^k) & \text{for } \widehat{S}, \\ \prod_{k \in \mathbb{N}} 2(1 + t^k) / (1 - t^k) & \text{for } \widehat{R}. \end{cases} \quad (3.50)$$

The factor of 2 for $\widehat{R}$ appears because of the relation

$$\dim V_j^{h,c} = (1 + 2\mu) \dim U(\widehat{Q}_-)_j. \quad (3.51)$$

For $\widehat{R}$ and $h = c/16$ we have for $\tilde{V} = \tilde{V}^{c/16,c} \equiv U(\tilde{R}_-)J_0v_0$

$$\text{ch } \tilde{V}(t) = t^{c/16} \prod_{k \in \mathbb{N}} (1 + t^k)/(1 - t^k). \quad (3.52)$$

We present now the character formulae for the irreducible GHWM L_Λ in the cases when $L_\Lambda \neq V^\Lambda$ (for $\widehat{W}$ cf. [216, 514, 157], (for partial results see [354, 353, 516, 362, 336])); for $\widehat{S}$, $\widehat{R}$ cf. [157], (for partial results see [354, 362, 276])).

In the N^0 case the embedded $V^{h+vmn,c}$ is irreducible while for $L^{h,c}$ using the results of [514] one can obtain

$$\text{ch } L^{h,c} = \text{ch } V^{h,c} - \text{ch } V^{h+vmn,c} = (1 - t^{vmn}) \text{ch } V^{h,c}. \quad (3.53)$$

In the N_-^1 case let us denote by $L_{0k}, L_{1k}, L'_{0k}, L'_{1k}$ the irreducible factor-modules of $V_{\ell k}, V'_{\ell k}$, respectively. Then we have ([216, 514], [157], formula (22)):

$$\begin{aligned} \text{ch } L_{\ell k} &= \text{ch } V_{\ell k} + \sum_{j>k} (\text{ch } V_{0j} + \text{ch } V_{1j}) \\ &\quad - \sum_{j \geq k} (\text{ch } V'_{0j} + \text{ch } V'_{1j}), \end{aligned} \quad (3.54a)$$

$$\begin{aligned} \text{ch } L'_{\ell k} &= \text{ch } V'_{\ell k} + \sum_{j>k} (\text{ch } V'_{0j} + \text{ch } V'_{1j}) \\ &\quad - \text{ch } V_{0j} - \text{ch } V_{1j}, \end{aligned} \quad (3.54b)$$

where in both formulae $\ell = 0, 1, k \in \mathbb{Z}_+$.

In the N_+^1 case we denote by $L_{\ell k}^+, L_{\ell k}'^+, (\ell = 0, 1)$, the irreducible factor-modules of $V_{\ell k}^+, V_{\ell k}'^+$, respectively. Then we have ([216, 514], [157], formula (35)):

$$\begin{aligned} \text{ch } L_{\ell k}^+ &= \text{ch } V_{\ell k}^+ + \sum_{j=1}^{k-1} (\text{ch } V_{0j} + \text{ch } V_{1j}) + \text{ch } L_{00}^+ \\ &\quad - \sum_{j=0}^{k-1} (\text{ch } V'_{0j} + \text{ch } V'_{1j}), \quad k > 0, \end{aligned} \quad (3.55a)$$

$$\begin{aligned} \text{ch } L_{\ell k}'^+ &= \text{ch } V_{\ell k}'^+ + \sum_{j=0}^{k-1} (\text{ch } V'_{0j} + \text{ch } V'_{1j}) - \text{ch } L_{00}^+ \\ &\quad - \sum_{j=1}^k (\text{ch } V_{0j} + \text{ch } V_{1j}), \quad k \geq 0, \end{aligned} \quad (3.55b)$$

where in both formulae $\ell = 0, 1$; ($\text{ch } L_{00}^+ = \text{ch } L_{10}^+ = \text{ch } V_{00}^+ = \text{ch } V_{10}^+$, since $V_{00}^+ = V_{10}^+$ is irreducible).

In the $N_{\pm}^{21}$ cases let us denote by $L_{\ell k}^0, L_{\ell k}^{+0}$ the irreducible factor module of $V_{\ell k}^0, V_{\ell k}^{+0}$, respectively. Then we have ([216, 514], [157], formulae (28) and (29)):

$$\begin{aligned} \text{ch } L_{0k}^0 &= \text{ch } V_{0k}^0 - \text{ch } V_{1k}^0, & k > 0, \\ \text{ch } L_{1k}^0 &= \text{ch } V_{1k}^0 - \text{ch } V_{0,k+1}^0, & k \geq 0, \end{aligned} \quad (3.56)$$

and ([216, 514], [157], formulae (42) and (43))

$$\begin{aligned} \text{ch } L_{0k}^{+0} &= \text{ch } V_{0k}^{+0} - \text{ch } V_{1,k-1}^{+0}, & k > 0, \\ \text{ch } L_{1k}^{+0} &= \text{ch } V_{1k}^{+0} - \text{ch } V_{0k}^{+0}, & k > 0, \end{aligned} \quad (3.57)$$

($\text{ch } L_{00}^{+0} = \text{ch } L_{10}^{+0} = \text{ch } V_{00}^{+0} = \text{ch } V_{10}^{+0}$, since $V_{00}^{+0} = V_{10}^{+0}$ is irreducible.) These formulae are valid also for the $R_{\pm}^{21}$ cases but all generalized Verma modules $V_{\ell k}^0, V_{\ell k}^{+0}$ should be replaced with the ordinary Verma modules $\bar{V}_{\ell k}^0, \bar{V}_{\ell k}^{+0}$ (with highest weight vector $J_0 v_0$ as discussed above).

In the $N_{\pm}^{22}, N_{\pm}^{23}$ cases let us denote by $L_k^{\epsilon}, L_k^{+\epsilon}$ the irreducible factor modules of $V_k^{\epsilon}, V_k^{+\epsilon}$, respectively. Then we have ([216, 514], [157], formula (31))

$$\text{ch } L_k^{\epsilon} = \text{ch } V_k^{\epsilon} - \text{ch } V_{k+1}^{\epsilon}, \quad k \geq 0, \quad (3.58)$$

and ([216, 514], [157], formulae (44) and (45))

$$\text{ch } L_k^{+\epsilon} = \text{ch } V_k^{+\epsilon} - \text{ch } V_{k-1}^{+\epsilon}, \quad k > 0. \quad (3.59)$$

($\text{ch } L_0^{+\epsilon} = \text{ch } V_0^{+\epsilon}$ since $V_0^{+\epsilon}$ is irreducible.)

3.5 Weyl group for Virasoro and $N = 1$ super-Virasoro algebras

In this section we follow [164]. Consider the following infinite abelian multiplicative group W_a [8] generated by the symbols $w(n), n \in \frac{1}{v}N$ with the properties

$$w(n)^2 = 1, \quad w(n)w(n') = w(n')w(n). \quad (3.60)$$

Thus the group W_a consists of the elements

$$w = w(n_1) \dots w(n_k), \quad k \in \mathbb{N}, \quad n_i \in \frac{1}{v}\mathbb{Z}_+, \quad w(0) \equiv 1. \quad (3.61)$$

Such a group is called a torsion group or a p -group [32] (here $p = 2$). If $w = 1$ or $w = w(n_1) \dots w(n_k), n_i \in \frac{1}{v}\mathbb{N}, n_i \neq n_j, i \neq j$ we shall say that w is given in a *reduced form*. Because of (3.60) it is clear that every element of W_a has a reduced form.

Let w be given in a reduced form, then the *length of w* , denoted $\ell(w)$ is defined as follows:

$$\ell(w) = \begin{cases} 0 & w = 1, \\ k & w = w(n_1) \dots w(n_k), \quad n_i \in \frac{1}{v}\mathbb{N}, \quad n_i \neq n_j, \quad i \neq j. \end{cases} \quad (3.62)$$

The length of any element w is defined as the length of its reduced form. (If $\ell(w) = k$, then w has $k!$ reduced forms.) We need to introduce the action of W_a on $\widehat{Q}_0^*$. We set for

w in a reduced form and $\lambda \in \widehat{Q}_0^*$:

$$\begin{aligned} 1 \cdot \lambda &= \lambda, \\ w \cdot \lambda &= \lambda + (n_1 + \cdots + n_k)\lambda_0, \\ w &= w(n_1) \dots w(n_k), \quad n_i \in \frac{1}{\nu}\mathbb{N}, \quad n_i \neq n_j, \end{aligned} \quad (3.63)$$

where $\lambda_0 \in \widehat{Q}_0^*$ is defined by

$$\lambda_0(L_0) = 1, \quad \lambda_0(z) = 0. \quad (3.64)$$

Setting as usual $\lambda(L_0) = h$, $\lambda(z) = c$ we shall write (3.63) equivalently as

$$\begin{aligned} 1 \cdot (h, c) &= (h, c), \\ w \cdot (h, c) &= (h + n_1 + \cdots + n_k, c) \equiv (w \cdot h, c). \end{aligned} \quad (3.65)$$

Note that this action is non-associative; however, this is not essential for our purposes.

Definition. Let $V = V^{h,c}$ be a (generalized) Verma module over $\widehat{Q}$. We shall say that an element $w \in W_a$ is *V-active* if either $w = 1$ or $w = w(n_1) \dots w(n_k)$, $n_i \in \frac{1}{\nu}\mathbb{N}$, $n_i \neq n_j$ and there exists a chain of invariant embeddings $V^{h,c} \rightarrow V_1 \rightarrow \cdots \rightarrow V_k$ (remember that the arrows point to the embedded modules), where the submodules V_j are isomorphic to (generalized) Verma modules over $\widehat{Q}$: $V_j \cong V^{h+n_1+\cdots+n_j,c}$, $1 \leq j \leq k$. For $h = c/16$ and $\widehat{R}$ we consider $\widehat{V}^{c/16,c}$ instead of $V^{c/16,c}$.

Obviously this definition can be extended to (generalized) HWM.

It is clear that V is irreducible iff only $w = 1$ is V-active. It is also clear that if w as above is V-active and $w \neq w(n)$ there are also other V-active elements of W_a .

Lemma 1. Let $w = w(n_1) \dots w(n_k)$ be V-active. Let $\ell \in \mathbb{N}$, $\ell < k$, $\{i_1, \dots, i_\ell\}$ be a subset of $\{n_1, \dots, n_k\}$, such that $i_1 < \cdots < i_\ell$, let $m_j = n_{i_{j-1}+1} + \cdots + n_{i_j}$, $j = 1, \dots, \ell$, $i_0 = 0$. Then $w' \equiv w(i_1) \dots w(i_\ell)$ is also V-active.

Proof. Let $V_j' = V^{h+m_1+\cdots+m_j,c}$. Note that $m_1 + \cdots + m_j = n_1 + \cdots + n_{i_j}$. Thus $V_j' = V_{i_j}$. Obviously $V \rightarrow V_{i_1} \rightarrow \cdots \rightarrow V_{i_\ell}$. $\square$

The usefulness of this notion should become clear from the following, which is the main result of this section.

Theorem 1. Let $L^{h,c}$ be an irreducible GHWM over $\widehat{Q}$, so that $L^{h,c} \cong V^{h,c}/I^{h,c}$, where $V \equiv V^{h,c}$ is the (generalized) Verma module of $\widehat{Q}$ with the same highest weight and $I^{h,c}$ is the maximal submodule of $V^{h,c}$. Then we have for the character of $L^{h,c}$

$$\text{ch } L^{h,c} = \sum_{\substack{w \in W_a \\ w \text{ V-active}}} (-1)^{\ell(w)} \text{ch } V^{w \cdot h, c}, \quad \text{or} \quad (3.66a)$$

$$\text{ch } L^{h,c}(t) = \text{ch } V^{h,c}(t) \sum_{\substack{w \in W_a \\ w \text{ V-active}}} (-1)^{\ell(w)} t^{w \cdot h - h}. \quad (3.66b)$$

Proof. First we note that if $V^{h,c}$ is irreducible $L^{h,c} = V^{h,c}$ and $\text{ch } L^{h,c} = \text{ch } V^{h,c}$ which follows also from (3.66) – the sum involves only $w = 1$ ($\ell(1) = 0$). Further we consider the different types of embeddings according to Table 3.1.

Case N^0 $V^{h+vmn,c}$ is irreducible, while for $V = V^{h,c}$ there are only two V -active elements $w = 1$ and $w = w(vmn)$. Thus we have from (3.66)

$$\text{ch } L^{h,c} = \text{ch } V^{h,c} - \text{ch } V^{h+vmn,c}, \quad (3.67)$$

which is the correct result according to (3.53).

Cases N_-^{22}, N_-^{23} We consider V_j^ϵ as given in Section 3.3 and Table 3.1. We shall not need the explicit values of h_j but only the fact that $h_j - h_i \in \frac{1}{v}\mathbb{N}$ for $i < j$. We introduce the following notation:

$$w_{ij} \equiv w(h_j - h_i), \quad i, j \in \mathbb{Z}_+, \quad i \leq j, \quad (w(0) \equiv 1). \quad (3.68)$$

It is clear that

$$w_{ij} \cdot h_\ell = h_\ell + h_j - h_i, \quad w_{ij} \cdot h_i = h_j. \quad (3.69)$$

We fix $k \in \mathbb{Z}_+$ and consider V_k^ϵ . It is clear that $W_{k,\ell}$, $\ell \in \mathbb{Z}_+$, $\ell \geq k$, is V_k^ϵ -active. We fix $\ell > k$. We want to enumerate all possible V_k^ϵ -active elements which act on h_k as $w_{k,\ell}$, i. e. $w_{k,\ell} \cdot h_k = h_\ell$. If $\ell = k + 1$ there are no other such elements. If $\ell > k + 1$ besides $w_{k,\ell}$ they are all of the form

$$w_{k,k_1} w_{k_1,k_2} \dots w_{k_r,\ell}, \quad r = 1, \dots, \ell - k - 1, \quad k < k_1 < \dots < k_r < \ell, \quad (3.70)$$

and we agree that $w_{k,\ell}$ is obtained for $r = 0$. For fixed r there are $\binom{\ell-k-1}{r}$ possible choices of $k_1, \dots, k_r$. We note that $\ell(w_{k,k_1}, \dots, w_{k_r,\ell}) = r + 1$. Thus for the character of $L_k^\epsilon = V_k^\epsilon / I_k^\epsilon$ we have

$$\begin{aligned} \text{ch } L_k^\epsilon(t) &= \left[1 + \sum_{\ell=k+1}^{\infty} \sum_{r=0}^{\ell-k-1} \sum_{k < k_1 < \dots < k_r < \ell} (-1)^{\ell(w_{k,k_1}, \dots, w_{k_r,\ell})} t^{w_{k,k_1} \dots w_{k_r,\ell} \cdot h - h} \right] \text{ch } V_k^\epsilon(t) \\ &= \left[1 + \sum_{\ell=k+1}^{\infty} t^{h_\ell - h_k} \sum_{r=0}^{\ell-k-1} (-1)^{r+1} \binom{\ell-k-1}{r} \right] \text{ch } V_k^\epsilon(t) \\ &= (1 - t^{h_{k+1} - h_k}) \text{ch } V_k^\epsilon(t) = \text{ch } V_k^\epsilon(t) - \text{ch } V_{k+1}^\epsilon(t), \end{aligned} \quad (3.71)$$

which coincides with the correct formula (3.58).

Cases N_-^{21}, R_-^{21} As is clear from Table 3.1 these cases can be treated as the previous cases, N_-^{22}, N_-^{23} . In the context of the theorem we only have to identify the modules V_{0k}^0 with V_{2k-1}^ϵ , $k \in \mathbb{N}$ and the modules V_{1k}^0 with V_{2k}^ϵ , $k \in \mathbb{Z}_+$. At the end we recover the two formulae (3.56) for $\text{ch } L_{0k}^0$, $k \in \mathbb{N}$ and $\text{ch } L_{1k}^0$, $k \in \mathbb{Z}_+$.

Case N_+^{22}, N_+^{23} We consider $V_j^{+\epsilon}$ as given in Section 3.3 and Table 3.1. Instead of (3.68) we introduce

$$w_{ij}^+ \equiv w(h_j^+ - h_i^+), \quad i, j \in \mathbb{Z}_+, \quad i \geq j, \quad (w(0) \equiv 1). \quad (3.72)$$

We apply the same reasoning as for the case N_-^{22}, N_-^{23} and obtain for $L_k^{+\epsilon} = V_k^{+\epsilon}/I_k^{+\epsilon}$, $k > 0$ (recalling that $L_0^{+\epsilon} = V_0^{+\epsilon}$)

$$\begin{aligned} \text{ch } L_k^{+\epsilon}(t) &= \left[1 + \sum_{\ell=0}^{k-1} \sum_{r=0}^{\ell-k-1} \sum_{k>k_1>\dots>k_r>\ell} (-1)^{\ell(w_{k,k_1}^+ \dots w_{k_r,\ell}^+)} t^{w_{k,k_1}^+ \dots w_{k_r,\ell}^+ \cdot h - h} \right] \text{ch } V_k^{+\epsilon}(t) \\ &= \left[1 + \sum_{\ell=0}^{k-1} t^{h_\ell^+ - h_k^+} \sum_{r=0}^{k-\ell-1} (-1)^{r+1} \binom{k-\ell-1}{r} \right] \text{ch } V_k^{+\epsilon}(t) \\ &= [1 - t^{h_{k-1}^+ - h_k^+}] \text{ch } V_k^{+\epsilon}(t) = \text{ch } V_k^{+\epsilon}(t) - \text{ch } V_{k-1}^{+\epsilon}(t), \quad k > 0, \end{aligned} \quad (3.73)$$

which coincides with the correct formula (3.59).

Cases N_+^{21}, R_+^{21} As is clear from Table 3.1 these cases can be treated as the previous cases N_+^{22}, N_+^{23} . In the context of the Theorem we only have to identify the modules V_{0k}^{+0} with V_{2k-1}^{ϵ} , $k \in \mathbb{N}$ and the modules V_{1k}^{+0} with V_{2k}^{ϵ} , $k \in \mathbb{Z}_+$. At the end we recover the two formulae (3.57) for $\text{ch } L_{0k}^0$, $k \in \mathbb{N}$ and $\text{ch } L_{1k}^0$, $k \in \mathbb{N}$. (We recall that $V_{10}^{+\epsilon}$ is irreducible.)

Case N_-^1 We consider $V_{0k}, V_{1k}, V'_{0k}, V'_{1k}$ from Table 3.1, given explicitly in (3.30), and we set

$$\begin{aligned} h_{0k} &= h + vk(\tilde{p}\tilde{q}k + \tilde{q}m - \tilde{p}n), & h_{1k} &= h + vk(\tilde{p}\tilde{q}k - \tilde{q}m + \tilde{p}n), \\ h'_{0k} &= h + v(\tilde{q}k + m)(\tilde{p}k + n), & h'_{1k} &= h + v(\tilde{q}k + \tilde{q} - m)(\tilde{p}k + \tilde{p} - n), \end{aligned} \quad (3.74)$$

with $h = h_{(m,n)}$ from (3.13).

We start with V_{0k} . Let $\mathcal{A}_{0\ell}^k$, $\ell > k$, $\mathcal{A}_{1\ell}^k$, $\ell > k$, $\mathcal{A}'_{0\ell}^k$, $\ell \geq k$, $\mathcal{A}'_{1\ell}^k$, $\ell \geq k$ denote the sets of V_{0k} -active elements of W_a such that when acting on h_{0k} we obtain $h_{0\ell}, h_{1\ell}, h'_{0\ell}, h'_{1\ell}$, respectively.

We claim that all elements of $\mathcal{A}_{0\ell}^k$ are obtained as follows. To every element $\tilde{w} \in \mathcal{A}_{0,\ell-1}^k$ there correspond three elements of $\mathcal{A}_{0\ell}^k$. First $w_1 = \tilde{w}w'$, where $w' \cdot h'_{0,\ell-1} = h_{0\ell}$. Next $\tilde{w} = \tilde{w}'w(u)$, where $\tilde{w}' \cdot h_{0k} = \tilde{h}$ (for $\ell = k+1$, $\tilde{w}' = 1$, $\tilde{h} = h_{0k}$), $w(u)\tilde{h} = h_{0,\ell-1}$. To $\tilde{w}$ there correspond $w_2 = \tilde{w}'w(u')$, where $w(u') \cdot \tilde{h} = h_{0\ell}$ and $w_3 = \tilde{w}'w(v)w(v')$, where $w(v) \cdot \tilde{h} = h'_{1,\ell-1}$, $w(v') \cdot h'_{1,\ell-1} = h_{0\ell}$. Thus $w_i \cdot h_{0k} = h_{0\ell}$, $\ell(w_1) = \ell(w_3) = \ell(\tilde{w}) + 1$, $\ell(w_2) = \ell(\tilde{w})$. Proceeding analogously we find that $|\mathcal{A}_{0\ell}^k| = 3^{2(\ell-k)-1}$, $|\mathcal{A}'_{0\ell}^k| = 3^{2(\ell-k)}$, $|\mathcal{A}_{1\ell}^k| = 3^{2(\ell-k)-1}$, $|\mathcal{A}'_{1\ell}^k| = 3^{2(\ell-k)}$. Further, let $\mathcal{A}_{0\ell}^{k+}$ respectively $\mathcal{A}_{0\ell}^{k-}$ (and analogously for the other A) be the elements of $\mathcal{A}_{0\ell}^k$ with even, respectively odd, length. (Naturally, $|\mathcal{A}_{0\ell}^{k+}| + |\mathcal{A}_{0\ell}^{k-}| = |\mathcal{A}_{0\ell}^k|$.)

We claim that

$$|A_{0\ell}^{k+}| = |A_{0\ell}^{k-}| + 1, \quad |A_{1\ell}^{k+}| = |A_{1\ell}^{k-}| + 1, \quad \ell > k, \quad (3.75a)$$

$$|A_{0\ell}'^{k+}| = |A_{0\ell}'^{k-}| - 1, \quad |A_{1\ell}'^{k+}| = |A_{1\ell}'^{k-}| - 1, \quad \ell \geq k. \quad (3.75b)$$

Indeed, let $|A_{0\ell}^{k+}| = |A_{0\ell}^{k-}| + 1$ for some ℓ . According to our reasoning above we have $|A_{0\ell}'^{k+}| = |A_{0\ell}^{k+}| + 2|A_{0\ell}^{k-}| = 3|A_{0\ell}^{k-}| + 1$, $|A_{0\ell}'^{k-}| = 2|A_{0\ell}^{k+}| + |A_{0\ell}^{k-}| = 3|A_{0\ell}^{k-}| + 2$. It remains to note that $|A_{0k+1}^{k+}| = 2$, $|A_{0k+1}^{k-}| = 1$.

Next we substitute (3.75) in (3.66b) and we obtain for L_{0k} (the irreducible quotient of V_{0k})

$$\begin{aligned} \text{ch } L_{0k}(t) &= \left[1 - \sum_{\ell \geq k} (t^{h_{0\ell}' - h_{0k}} + t^{h_{1\ell}' - h_{0k}}) \right. \\ &\quad \left. + \sum_{\ell > k} (t^{h_{0\ell} - h_{0k}} + t^{h_{1\ell} - h_{0k}}) \right] \text{ch } V_{0k}(t) \end{aligned} \quad (3.76a)$$

$$\begin{aligned} &= \text{ch } V_{0k}(t) - \sum_{\ell \geq k} (\text{ch } V_{0\ell}'(t) + \text{ch } V_{1\ell}'(t)) \\ &\quad + \sum_{\ell > k} (\text{ch } V_{0\ell}(t) + \text{ch } V_{1\ell}(t)), \end{aligned} \quad (3.76b)$$

which coincides with the correct result (3.54a).

For $\text{ch } L_{1k}$, $\text{ch } L_{0k}'$, $\text{ch } L_{1k}'$ the argument goes analogously and the correct results in (3.54) are obtained as claimed from (3.66).

Case N_+^1 We consider V_{0k}^+ , V_{1k}^+ , $V_{0k}'^+$, $V_{1k}'^+$ from Table 3.1 given explicitly by (3.30) with the changes $\tilde{p} \rightarrow -\tilde{p}$, $m \rightarrow -m$ (as explained before equation (3.55)). The proof is analogous to that of Case N_-^1 . We start with V_{0k} , $k > 0$. Let $\tilde{\mathcal{A}}_{0\ell}^k$, $\ell < k$, $\tilde{\mathcal{A}}_{1\ell}^k$, $0 < \ell < k$, $\tilde{\mathcal{A}}_{0\ell}'^k$, $\ell < k$, $\tilde{\mathcal{A}}_{1\ell}'^k$, $\ell < k$ denote the sets of V_{0k} -active elements of W_a such that when acting on h_{0k}^+ we obtain $h_{0\ell}^+$, $h_{1\ell}^+$, $h_{0\ell}'^+$, $h_{1\ell}'^+$, respectively. Analogously to the N_-^1 case we find that $|\tilde{\mathcal{A}}_{1\ell}^k| = 3^{2(k-\ell)-1}$, $|\tilde{\mathcal{A}}_{0\ell}'^k| = 3^{2(k-\ell-1)}$, $|\tilde{\mathcal{A}}_{1\ell}^k| = 3^{2(k-\ell)-1}$, $|\tilde{\mathcal{A}}_{0\ell}^k| = 3^{2(k-\ell-1)}$. For the elements with even (respectively odd) length $\tilde{\mathcal{A}}_{0\ell}^{k+}$ (respectively $\tilde{\mathcal{A}}_{0\ell}^{k-}$), we obtain the same relations as (3.75):

$$|\tilde{\mathcal{A}}_{0\ell}^{k+}| = |\tilde{\mathcal{A}}_{0\ell}^{k-}| + 1, \quad \ell < k, \quad |\tilde{\mathcal{A}}_{1\ell}^{k+}| = |\tilde{\mathcal{A}}_{1\ell}^{k-}| + 1, \quad 0 < \ell < k, \quad (3.77a)$$

$$|\tilde{\mathcal{A}}_{0\ell}'^{k+}| = |\tilde{\mathcal{A}}_{0\ell}'^{k-}| - 1, \quad |\tilde{\mathcal{A}}_{1\ell}'^{k+}| = |\tilde{\mathcal{A}}_{1\ell}'^{k-}| - 1, \quad \ell < k. \quad (3.77b)$$

Then we substitute (3.77) in (3.66b) and obtain (3.55a) for $\text{ch } L_{0k}^+$. Analogously one obtains the rest of (3.55).

This concludes the proof of the theorem. $\square$

3.6 Fock modules over the Virasoro algebra

In this section we follow mainly [109]. We start with an infinite system of oscillators $a_k, k \in \mathbb{Z}$, which form a Lie algebra $\mathcal{A}$ satisfying

$$[a_k, a_\ell] = k\delta_{k, -\ell}. \quad (3.78)$$

The Fock module uses the universal enveloping algebra of $\mathcal{A}$ with the basis:

$$a_{k_1} \dots a_{k_\ell}, \quad k_1 \leq \dots \leq k_\ell. \quad (3.79)$$

For more compact expressions one uses the so-called ‘normal ordering’

$$: a_k a_\ell : \equiv \begin{cases} a_k a_\ell & \text{if } k \leq \ell, \\ a_\ell a_k & \text{if } k > \ell. \end{cases} \quad (3.80)$$

Then the Virasoro generators may be realized through the oscillators as follows:

$$L_n = \frac{1}{2} \sum_{k \in \mathbb{Z}} : a_k a_{n-k} : - (n+1)\xi a_n \quad (3.81)$$

where ξ is called the ‘background charge’ in the applications. From (3.81) and (3.78) easily follows:

$$[L_n, a_k] = -ka_{n+k} - n(n+1)\xi\delta_{n, -k}. \quad (3.82)$$

Using (3.81) one recovers the Virasoro algebra (3.1) assuming that the central charge operator $\hat{c}$ is replaced by its eigenvalue c and that c and ξ are related as follows:

$$c = 1 - 12\xi^2. \quad (3.83)$$

We introduce the level of the basis monomials as for Verma modules, namely as minus the corresponding eigenvalue of the commutation with L_0

$$[L_0, a_{k_1} \dots a_{k_\ell}] = (k_1 + \dots + k_\ell) a_{k_1} \dots a_{k_\ell} \quad (3.84)$$

i. e. the level of $a_{k_1} \dots a_{k_\ell}$ is $-(k_1 + \dots + k_\ell)$.

Now a Fock module $F(\xi, \eta)$ is determined by two parameters, one of which is ξ , and the other is denoted by η (called ‘shifted momentum’ in the applications), and by a vacuum vector $|\eta\rangle$, so that:

$$a_k |\eta\rangle = 0, \quad k > 0, \quad a_0 |\eta\rangle = (\eta + \xi) |\eta\rangle. \quad (3.85)$$

As a vector space the Fock module $F(\xi, \eta)$ has the basis:

$$a_{k_1} \dots a_{k_\ell} |\eta\rangle, \quad k_1 \leq \dots \leq k_\ell < 0. \quad (3.86)$$

In order to find the action of the Virasoro algebra on the Fock modules one uses:

$$L_n|\eta\rangle = 0, \quad n > 0, \quad L_0|\eta\rangle = \frac{1}{2}(\eta^2 - \xi^2)|\eta\rangle, \quad (3.87)$$

and (3.82).

Thus, we have a highest weight module over the Virasoro algebra with

$$h = \frac{1}{2}(\eta^2 - \xi^2), \quad c = 1 - 12\xi^2. \quad (3.88)$$

The Fock module $F(\xi, \eta)$ is reducible under the same condition (3.13) as the Verma module $V_{\frac{1}{2}(\eta^2 - \xi^2), 1 - 12\xi^2}$. We shall also use the notation $\eta_{(m,n)}$ to write

$$h = h_{(m,n)} = \frac{1}{2}(\eta_{(m,n)}^2 - \xi^2). \quad (3.89)$$

But Fock modules are quite different from Verma modules; cf. [214]. As a first step for the study of Fock modules we look for singular vectors. Since these will be given in terms of Schur polynomials we need some elementary notions as regards the latter.

Let us recall (cf., e. g. [533]) that the elementary Schur polynomials S_k are defined by the exponential generating function:

$$\exp\left(\sum_{k \in \mathbb{N}} t^k x_k\right) = \sum_{k \in \mathbb{Z}} t^k S_k(x), \quad x = (x_1, x_2, \dots), \quad (3.90)$$

which implies

$$S_k(x) = 0, \quad k < 0, \quad S_0(x) = 1, \quad (3.91a)$$

$$S_k(x) = \sum_{k_1 + 2k_2 + \dots = k} \frac{x_1^{k_1} x_2^{k_2}}{k_1! k_2!} \dots, \quad k > 0. \quad (3.91b)$$

Further with any partition $\lambda = \{\lambda_1 \geq \lambda_2 \geq \dots\}$ one associates a Schur polynomial:

$$S_\lambda(x) = S_{\lambda_1, \lambda_2, \dots}(x) = \det(S_{\lambda_i + k - j}(x))_{j,k}. \quad (3.92)$$

For a given partition $\lambda = \{\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_\ell\}$ with $\lambda_\ell > 0$ we shall call ℓ the *length* of λ .

Next we introduce variables in terms of which we shall write the singular vectors. Namely, for $k \neq 0$ we denote $x_k = \gamma a_{-k}/k$, $\gamma = \gamma_\pm = \sqrt{2}\alpha_\pm$, then we have

$$\begin{aligned} [L_n, x_k] &= \frac{\gamma}{k} [L_n, a_{-k}] = \gamma(a_{n-k} - (k+1)\xi\delta_{n,k}) \\ &= \begin{cases} (k-n)x_{k-n} & n \neq k, \\ \gamma(a_0 - (k+1)\xi) & n = k. \end{cases} \end{aligned} \quad (3.93)$$

Further, we shall write a unified formula assuming that $((k-n)x_{k-n})|_{k=n} = 0$ and substituting the central oscillator a_0 with its value in the Fock module, namely,

with $\xi + \eta$. Thus, we get

$$[L_n, x_k] = (k - n)x_{k-n} + \gamma(\eta - k\xi)\delta_{n,k}. \quad (3.94)$$

For the derivation of the singular vectors we need only the action of L_1 and L_2 (since they generate all L_n , $n \geq 3$). Consider a function $f(x)$ of $x = (x_1, x_2, \dots)$. Then from (3.94) we obtain at once

$$[L_1, f(x)] = \sum_{j=1}^{\infty} jx_j \frac{\partial f}{\partial x_{j+1}} + \sqrt{2}\gamma(\eta - \xi) \frac{\partial f}{\partial x_1}, \quad (3.95)$$

and after some manipulation

$$[L_2, f(x)] = \sum_{j=1}^{\infty} jx_j \frac{\partial f}{\partial x_{j+2}} + \sqrt{2}\gamma(\eta - 2\xi) \frac{\partial f}{\partial x_2} + \gamma^2 \frac{\partial^2 f}{\partial x_1^2} + 2\gamma^2 \frac{\partial f}{\partial x_1} \frac{\partial}{\partial x_1}. \quad (3.96)$$

The last term in (3.96) indicates that L_2 is not a derivative operator on functions of x . We need the following properties of the elementary Schur polynomials:

$$\frac{\partial S_k(x)}{\partial x_\ell} = S_{\ell-k}, \quad \sum_{\ell=j+1}^k (\ell - j)x_{\ell-j} S_{k-\ell} = (k - j)S_{k-j}. \quad (3.97)$$

Using these properties we obtain from (3.95) and (3.96) the formulae necessary for the derivation of the singular vectors:

$$[L_1, S_k(x)] = (k - 1 + \gamma(\eta - \xi))S_{k-1}(x), \quad (3.98)$$

$$[L_2, S_k(x)] = (k - 1 + \gamma(\eta - \xi))S_{k-2}(x) + \gamma^2 S_{k-1} \frac{\partial}{\partial x_1}. \quad (3.99)$$

We are now ready to tackle the question of the explicit expressions for the singular vectors. As in the case of Verma modules the singular vector of level t is given by a linear combination of basis monomials of the oscillator algebra of level t applied to the vacuum. In terms of the x -variables we use (3.94) for $n = 0$,

$$[L_0, x_k] = kx_k, \quad (3.100)$$

and thus we get

$$[L_0, S_k] = kS_k, \quad [L_0, S_\lambda] = (\lambda_1 + \dots + \lambda_\ell)S_\lambda \quad (3.101)$$

(ℓ is the length of λ).

We consider first the case $c = 1$, i. e. $\xi = 0$. The singular vector of level mn is given by [533]

$$v_{m,n}^{c=1} = S_{\underbrace{n, \dots, n}_m} \left(\frac{\sqrt{2}}{k} a_{-k} \right) |\eta\rangle, \quad (3.102a)$$

or, equivalently, by

$$v_{m,n}^{c=1} = S_{\underline{m}, \dots, \underline{m}} \left(-\frac{\sqrt{2}}{k} a_{-k} \right) |\eta\rangle, \quad (3.102b)$$

the two expressions differing by sign. Let us note that the Schur polynomials involved are exactly of length m (respectively n), i. e. each term involves the product of exactly m (respectively n), elementary Schur polynomials.

These formulae were generalized for the discrete cases $h = h_{(m,n)}^-$ and $c = c_{p,q}^- < 1$ in [109]. For $m = 1$ or $n = 1$ one may check that the singular vectors are given by

$$v_{1,n} = S_n \left(\frac{\sqrt{2}}{k} \alpha_+ a_{-k} \right) |\eta\rangle, \quad m = 1, \quad (3.103a)$$

$$v_{m,1} = S_m \left(\frac{\sqrt{2}}{k} \alpha_- a_{-k} \right) |\eta\rangle, \quad n = 1, \quad (3.103b)$$

which reduce to (3.102) for $c = 1, p = q = 1, \alpha_{\pm} = \pm 1$. To check (3.103a) one uses (3.98) and (3.99) to obtain

$$[L_{\ell}, S_k] = (k - n) S_{k-\ell} \Rightarrow L_{\ell} S_n(x) |\eta\rangle = 0, \quad \ell = 1, 2, \quad (3.104)$$

the latter equality meaning that $S_n(x) |\eta\rangle$ is a singular vector. Similarly one checks (3.103b).

The cases $m, n > 1$ are much more interesting. In particular, for $m = 2$ or $n = 2$ we have

$$v_{m=2,n} = \sum_{k=0}^n \beta_k^n(\alpha_+) S_{2n-k,k} \left(\frac{\sqrt{2}}{j} \alpha_+ a_{-j} \right) |\eta\rangle, \quad (3.105a)$$

$$v_{m,n=2} = \sum_{k=0}^m \beta_k^m(\alpha_-) S_{2m-k,k} \left(\frac{\sqrt{2}}{j} \alpha_- a_{-j} \right) |\eta\rangle, \quad (3.105b)$$

$$\begin{aligned} \beta_k^t(\alpha_{\pm}) &= \frac{(-1)^k \Gamma(2(t + u_{\pm}))}{\Gamma(2t - k + u_{\pm} + 1) \Gamma(k + u_{\pm})} \\ &= (-1)^k \binom{2(t + u_{\pm}) - 1}{k + u_{\pm} - 1}, \end{aligned} \quad (3.105c)$$

where $u_{\pm} = \gamma_{\pm}(\eta - \xi) - 1 = \alpha_{\pm}^2 - t$, and only the expression in terms of Γ -functions can be used if $u_{\pm}$ is not integer.

Our first observation is that in (3.105), all Schur polynomials of level $2t$ ($t = m$ or $t = n$) and of length ≤ 2 are involved, since the term with $k = 0$ involves the elementary Schur polynomial $S_{2t} = S_{2t,0}$. Of course, there are partial cases when not all terms are present in (3.105a,b). Thus, for $p = 1$, respectively $q = 1$, we have $u_+ = q - n$ for (3.105a), respectively $u_- = p - m$ for (3.105b), and $\beta_k^t(\alpha_{\pm}) = 0$ for $k < 1 - u_{\pm}$. Finally, for $c = 1, p = q = 1$ we have $\beta_k^t(\pm 1) = 0$ for $k < t$ and (3.105a,b) collapse (up to signs) to (3.102a,b). Note that $\beta_i^t \neq 0$ in all cases.

Let us give also the example of a singular vector in the case $m = n = 3$:

$$\begin{aligned}
 v_{3,3} &= \sum_{9-j-s \geq j \geq s \geq 0} \beta_{j,s}^9(\alpha_+) S_{9-j-s,j,s} \left(\frac{\sqrt{2}}{k} \alpha_+ a_{-k} \right) |\eta\rangle, \\
 \beta_{j,s}^9(\alpha_+) &= \frac{(-1)^{j+s} \beta_{j,s}}{\Gamma(10-j-s+u)\Gamma(j+u)\Gamma(s+u-1)u}, \\
 \beta_{0,0} &= \beta_{1,0} = (u-3)(u+8), \quad \beta_{1,1} = \beta_{2,1} = (u-1)(u+6), \\
 \beta_{4,0} &= \beta_{4,1} = -(u+2)(u+3), \quad \beta_{2,2} = \beta_{3,2} = \beta_{3,3} = u(u+5), \\
 \beta_{2,0} &= -18, \quad \beta_{3,1} = -6, \quad \beta_{3,0} = -(u^2 + 5u + 12),
 \end{aligned} \tag{3.106}$$

where $u = u_+ = \gamma_+(\eta_{(3,3)} - \xi) - 1 = 2\alpha_+^2 - 3 = 2q/p - 3 > -3$. Note that $\beta_{j,s}^9$ have no poles (even for $u = 0$). For $c = 1$, $p = q = 1 = -u$, all coefficients are zero except $\beta_{3,3}^9$ and (3.106) goes into (3.103). For $u = -2, 0, 1, 3$ there are also some vanishing coefficients (always $\beta_{0,0}^9$). Note that $\beta_{3,3}^9 \neq 0$ in all cases. The same expression is valid if we replace $\alpha_+ \rightarrow \alpha_-$, $u = u_+ \rightarrow u_- = 2\alpha_-^2 - 3 = 2p/q - 3$.

Thus we are lead to the conjecture that the general expression for the singular vectors should be

$$\begin{aligned}
 v_{m,n} &= \sum_{\substack{k_1 \geq \dots \geq k_m \geq 0 \\ k_1 + \dots + k_m = mn}} \frac{(-1)^{k_1} \beta_{k_1, \dots, k_m}(\alpha_+)}{\Gamma(k_1 + u_+ + 1) \dots \Gamma(k_m + u_+ - m + 2)} \\
 &\quad \times S_{k_1, \dots, k_m} \left(\frac{\sqrt{2}}{j} \alpha_+ a_{-j} \right) |\eta\rangle
 \end{aligned} \tag{3.107a}$$

or, up to sign,

$$\begin{aligned}
 v_{m,n} &= \sum_{\substack{k_1 \geq \dots \geq k_n \geq 0 \\ k_1 + \dots + k_n = mn}} \frac{(-1)^{k_1} \beta_{k_1, \dots, k_n}(\alpha_-)}{\Gamma(k_1 + u_- + 1) \dots \Gamma(k_m + u_- - n + 2)} \\
 &\quad \times S_{k_1, \dots, k_n} \left(\frac{\sqrt{2}}{j} \alpha_- a_{-j} \right) |\eta\rangle
 \end{aligned} \tag{3.107b}$$

where $u_{\pm} = \gamma_{\pm}(\eta_{(m,n)} - \xi) - 1 = \sqrt{2}\alpha_{\pm}(\eta_{(m,n)} - \xi) - 1$, and the coefficients obey a simple recursion relation (from the action of L_1):

$$\beta_{k_1, \dots, k_n} = \sum_{\ell=2}^n \beta_{k_1-1, \dots, k_{\ell}+1, \dots, k_n}, \tag{3.108}$$

and a more complicated one (from the action of L_2), which we omit. Note that in (3.107a) (respectively, (3.107b)), $n \leq k_1 \leq mn$ (respectively $m \leq k_1 \leq mn$), $0 \leq k_m \leq n$ (respectively $0 \leq k_n \leq m$), $\beta_{n, \dots, n} \neq 0$, $\beta_{m, \dots, m} \neq 0$. Thus according to our conjecture Schur polynomials of level mn , and of arbitrary length $\leq m$ (or $\leq n$), are involved, i. e. every term involves the product of at most m (respectively n), elementary Schur polynomials.

3.7 Characters of the unitarizable highest weight modules over the $N = 2$ superconformal algebras

In this section we follow mainly [79] (by Boucher–Friedan–Kent) and [158].

3.7.1 Preliminaries

The $N = 2$ superconformal algebras (or super-Virasoro algebras) in two dimensions are three complex Lie superalgebras: the $N = 2$ Neveu–Schwarz superalgebra [1], the $N = 2$ Ramond superalgebra [1], the $N = 2$ twisted superalgebra [79], which are denoted as $\mathcal{A}$, $\mathcal{P}$, $\mathcal{T}$, respectively, or $\mathcal{G}$ when a statement holds for all three superalgebras. They have the following nontrivial super-Lie brackets:

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{1}{4}z(m^3 - m)\delta_{m,-n} \quad (3.109a)$$

$$[L_m, G_n^j] = \left(\frac{1}{2}m - n\right)G_{m+n}^j, \quad j = 1, 2 \quad (3.109b)$$

$$[L_m, Y_n] = -nY_{m+n}, \quad [Y_m, Y_n] = zm\delta_{m,-n} \quad (3.109c)$$

$$[Y_m, G_n^j] = i\epsilon^{jk}G_{m+n}^k, \quad \epsilon^{jk} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (3.109d)$$

$$[G_m^j, G_n^k]_+ = 2\delta^{jk}L_{m+n} + i\epsilon^{jk}(m - n)Y_{m+n} + z\left(m^2 - \frac{1}{4}\right)\delta^{jk}\delta_{m,-n} \quad (3.109e)$$

where $m \in \mathbb{Z}$ in L_m for all superalgebras; $m \in \mathbb{Z}$ in Y_m and $m \in \frac{1}{2} + \mathbb{Z}$ in G_m^j for $\mathcal{A}$; $m \in \mathbb{Z}$ in Y_m and G_m^j for $\mathcal{P}$; $m \in \mathbb{Z}$ in G_m^1 and $m \in \frac{1}{2} + \mathbb{Z}$ in Y_m and G_m^2 for $\mathcal{T}$.

The standard triangular decomposition of $\mathcal{G}$ is

$$\mathcal{G} = \mathcal{G}_+ \oplus \mathcal{H} \oplus \mathcal{G}_- \quad (3.110a)$$

$$\mathcal{H} = \tilde{\mathcal{L}}_q^*\{z, L_0, Y_0\} \quad \text{for } \mathcal{A}, \mathcal{P},$$

$$= \tilde{\mathcal{L}}_q^*\{z, L_0, G_0^1\} \quad \text{for } \mathcal{T}, \quad (G_0^1)^2 = L_0 - z/8 \quad (3.110b)$$

$$\mathcal{G}_+ = \tilde{\mathcal{L}}_q^*\{L_m, m > 0, Y_n, n > 0, G_p^j, p > 0\} \oplus \tilde{\mathcal{L}}_q^*\{\tilde{G}_0\}_{\mathcal{P}} \quad (3.110c)$$

$$\mathcal{G}_- = \tilde{\mathcal{L}}_q^*\{L_m, m < 0, Y_n, n < 0, G_p^j, p < 0\} \oplus \tilde{\mathcal{L}}_q^*\{G_0\}_{\mathcal{P}} \quad (3.110d)$$

where the generators $G_0, \tilde{G}_0$ which appear for $\mathcal{P}$ in (3.110c,d) are the zero modes of

$$G_n = \frac{1}{2}(G_n^1 + iG_n^2), \quad \tilde{G}_n = \frac{1}{2}(G_n^1 - iG_n^2). \quad (3.111)$$

3.7.2 Highest weight modules, reducibility, unitarity

A highest weight module (HWM) over $\mathcal{G}$ is characterized by its highest weight $\lambda \in \mathcal{H}^*$ and highest weight vector v_0 so that $Xv_0 = 0$, for $X \in \mathcal{G}_+$, $Hv_0 = \lambda(H)v_0$ for $H \in \mathcal{H}$. Denote $\lambda(L_0) = h$, $\lambda(z) = c$, $\lambda(Y_0) = q$. [Note that interchanging G_0 and $\tilde{G}_0$ in (3.110c,d)

means to pass from P^+ to P^- modules in the terminology of [79].] The largest HWM with these properties is the Verma module $V^\lambda = V^{h,c,q}$ ($= V^{h,c}$ for $\mathcal{T}$), which is isomorphic to $U(\mathcal{G}_-)v_0$, where $U(\mathcal{G}_-)$ denotes the universal enveloping algebra of $\mathcal{G}_-$. Denote by L^λ (respectively $L^{h,c,q}$, $L^{h,c}$) the factor module V^λ/I^λ , where I^λ is the maximal proper submodule of V^λ . Then every irreducible HWM over $\mathcal{G}$ is isomorphic to some L^λ .

A Verma module $V^{h,c,q}$ ($V^{h,c}$) over $\mathcal{G}$ is reducible if and only if [79]:

$$f_{r,s}^A \equiv 2h(c-1) - q^2 - \frac{1}{4}(c-1)^2 + \frac{1}{4}[(c-1)r+s]^2 = 0, \\ \text{for some } r \in \mathbb{N}, s \in 2\mathbb{N}, \quad (3.112a)$$

$$\text{or } g_n^A \equiv 2h - 2nq + (c-1)\left(n^2 - \frac{1}{4}\right) = 0, \\ \text{for some } n \in \frac{1}{2} + \mathbb{Z}, \quad \text{for } \mathcal{A}; \quad (3.112b)$$

$$f_{r,s}^P \equiv 2(c-1)\left(h - \frac{1}{8}\right) - q^2 + \frac{1}{4}[(c-1)r+s]^2 = 0, \\ \text{for some } r \in \mathbb{N}, s \in 2\mathbb{N}, \quad (3.112c)$$

$$\text{or } g_n^P \equiv 2h - 2nq + (c-1)\left(n^2 - \frac{1}{4}\right) - \frac{1}{4} = 0, \\ \text{for some } n \in \mathbb{Z}, \quad \text{for } \mathcal{P}; \quad (3.112d)$$

$$f_{r,s}^T \equiv 2(c-1)\left(h - \frac{1}{8}\right) + \frac{1}{4}[(c-1)r+s]^2 = 0, \\ \text{for some } r \in \mathbb{N}, s \in 2\mathbb{N} - 1, \quad \text{for } \mathcal{T}. \quad (3.112e)$$

The necessary conditions for the unitarity of $L^{h,c,q}$ ($L^{h,c}$) are [79]

$$\text{case } A_3: \quad c \geq 1, g_n^A \geq 0, \quad \text{for all } n \in \frac{1}{2} + \mathbb{Z}; \quad (3.113a)$$

$$\text{case } A_2: \quad c \geq 1, f_{1,2}^A \geq 0, g_n^A = 0, g_{n+\text{sign}(n)}^A \leq 0, \\ \text{for some } n \in \frac{1}{2} + \mathbb{Z}; \quad (3.113b)$$

$$\text{case } A_0: \quad c < 1, c = 1 - \tilde{\omega}, \quad h = \frac{1}{m}\left(jk - \frac{1}{4}\right), \quad q = \frac{1}{m}(j-k), \\ \text{for } m \in 1 + \mathbb{N}, j, k \in \frac{1}{2} + \mathbb{Z}, \\ 0 < j, k, j+k \leq m-1; \quad (3.113c)$$

$$\text{case } P_3: \quad c \geq 1, g_n^P \geq 0, \quad \text{for all } n \in \mathbb{Z}; \quad (3.114a)$$

$$\text{case } P_2: \quad c \geq 1, f_{1,2}^P \geq 0, g_n^P = 0, g_{n+\text{sign}(n)}^P < 0, \\ \text{for some } n \in \mathbb{Z}, \quad \text{sign}(0) = \pm 1 \text{ for } P^\pm; \quad (3.114b)$$

$$\text{case } P_0: \quad c < 1, c = 1 - \tilde{\omega}, \quad h = \frac{1}{8}c + \frac{jk}{m}, \quad q = \pm \frac{1}{m}(j-k), \\ \text{for } m \in 1 + \mathbb{N}, j, k \in \mathbb{Z}, \\ 0 \leq j-1, k, j+k \leq m-1; \quad (3.114c)$$

$$\text{case } T_2: \quad c \geq 1, \quad h \geq \frac{1}{8}c; \quad (3.115a)$$

$$\begin{aligned} \text{case } T_0: \quad c < 1, \quad c = 1 - \tilde{\omega}, \quad h = \frac{1}{8}c + \frac{1}{16m}(m - 2r)^2, \\ \text{for } m \in 1 + \mathbb{N}, \quad r \in \mathbb{N}, \quad 1 \leq r \leq \frac{1}{2}m. \end{aligned} \quad (3.115b)$$

Furthermore, we write $V^{h,c,(q)}, L^{h,c,(q)}$ in the cases when a statement holds for $V^{h,c,q}, L^{h,c,q}$ over $\mathcal{A}, \mathcal{P}$ as written and for $V^{h,c}, L^{h,c}$ over $\mathcal{T}$ after deleting q and all related quantities.

3.7.3 Character formulae

The weight decomposition of $V^{h,c,(q)}$ is

$$V^{h,c,(q)} = \bigoplus_{n,(m)} V_{n,(m)}^{h,c,(q)}, \quad (3.116a)$$

$$\begin{aligned} V_{n,(m)}^{h,c,(q)} &= \{v \in V^{h,c,(q)} \mid L_0 v = (h + n)v, \text{ for } \mathcal{G}, \\ Y_0 v &= (q + m)v, \text{ for } \mathcal{A}, \mathcal{P}\}, \end{aligned} \quad (3.116b)$$

where the ranges of n, m in (3.116) are

$$\begin{aligned} n &\in \frac{1}{2}\mathbb{Z}_+, \quad m \in 2n + 2\mathbb{Z}, \quad |m| \leq \sqrt{2n}, \quad \text{for } \mathcal{A}, \\ n &\in \mathbb{Z}_+, \quad m \in \mathbb{Z}, \quad \frac{1}{2}(1 - \sqrt{8n+1}) \leq m \leq \frac{1}{2}(1 + \sqrt{8n+1}), \quad \text{for } \mathcal{P}, \\ n &\in \frac{1}{2}\mathbb{Z}_+, \quad \text{for } \mathcal{T}. \end{aligned} \quad (3.117)$$

n is called the level of $V_{n,(m)}^{h,c,(q)}$, m its relative charge.

Then the character of $V^{h,c,(q)}$ may be defined as follows [79]:

$$\begin{aligned} \text{ch } V^{h,c,q} &= \sum_{n,m} (\dim V_{n,m}^{h,c,q}) x^{h+n} y^{q+m} = \sum_{n,m} P(n, m) x^{h+n} y^{q+m} \\ &= x^h y^q \psi(x, y); \end{aligned} \quad (3.118a)$$

$$\begin{aligned} \text{ch } V^{h,c} &= \sum_n (\dim V_n^{h,c}) x^{h+n} = \sum_n P_T(n) x^{h+n} \\ &= x^h \psi_T(x); \end{aligned} \quad (3.118b)$$

$$\begin{aligned} \psi_A(x, y) &\equiv \sum_{n,m} P_A(n, m) x^n y^m \\ &= \prod_{k \in \mathbb{N}} \frac{(1 + x^{k-1/2} y)(1 + x^{k-1/2} y^{-1})}{(1 - x^k)^2}; \end{aligned} \quad (3.119a)$$

$$\begin{aligned}\psi_P(x, y) &\equiv \sum_{n, m} P_P(n, m) x^n y^{m-1/2} \\ &= (y^{1/2} + y^{-1/2}) \prod_{k \in \mathbb{N}} \frac{(1 + x^k y)(1 + x^k y^{-1})}{(1 - x^k)^2};\end{aligned}\quad (3.119b)$$

$$\psi_T(x) \equiv \sum_n P_T(n) x^n = \prod_{k \in \mathbb{N}} \frac{(1 + x^k)(1 + x^{k-1/2})}{(1 - x^k)(1 - x^{k-1/2})}.\quad (3.119c)$$

For P^- representations one should write $y^{m+1/2}$ instead of $y^{m-1/2}$ [79].

Proposition 1 ([79, 158]). *The character formulae for the unitary cases $A_3(P_3)$, with either $c > 1$ and $g_n > 0$, $\forall n \in \frac{1}{2} + \mathbb{Z}$ ($\forall n \in \mathbb{Z}$), or $c = 1$, and cases T_2 are given by*

$$\text{ch } L^{h, c, q} = \text{ch } V^{h, c, q} \quad (3.120a)$$

$$\text{ch } L^{h, c} = \text{ch } V^{h, c}, \quad h \neq \frac{c}{8}, \quad \text{ch } L^{\frac{c}{8}, c} = \frac{1}{2} \text{ch } V^{\frac{c}{8}, c}.\quad (3.120b)$$

Note that the Verma modules involved are irreducible except in the last case, where $V_{\frac{c}{8}, c}^{\frac{c}{8}, c} = I_{\frac{c}{8}, c}^{\frac{c}{8}, c} \oplus V_{\frac{c}{8}, c}^{\frac{c}{8}, c} / I_{\frac{c}{8}, c}^{\frac{c}{8}, c}$, $I_{\frac{c}{8}, c}^{\frac{c}{8}, c} \cong V_{\frac{c}{8}, c}^{\frac{c}{8}, c} / I_{\frac{c}{8}, c}^{\frac{c}{8}, c}$. $\diamond$

Proposition 2 ([158]). *The character formulae for the unitary cases $A_3(P_3)$, with $c > 1$, $q/(c-1) = n_0 \in \frac{1}{2} + \mathbb{Z}$ ($n_0 \in \mathbb{Z}$), and $g_{n_0} = 0$, and for the cases $A_2(P_2)$, with $f_{1,2} > 0$, are given by*

$$\text{ch } L^{h, c, q} = \widetilde{\text{ch}}_n V^{h, c, q} \equiv \frac{1}{(1 + x^{|n|} y^{\text{sign}(n)})} \text{ch } V^{h, c, q} \quad (3.121)$$

where for A_3, P_3 , $n = n_0$, and for A_2, P_2 , n is such that $g_n = 0$, $g_{n+\text{sign}(n)}^A < 0$.

Proof. Actually, the proposition holds in a more general situation beyond the unitary cases, namely, when, for a fixed $V^{h, c, q}$ ((3.112b) ((3.112d))) holds for some n , possibly also for some n' such that $\text{sign}(n) = \text{sign}(n')$ and $|n'| > |n|$, and (3.112a) ((3.112c)) does not hold for any r, s . [In the statement of Proposition 2 the additional reducibility appears in the cases $A_2(P_2)$ when $2q(c-1) \in \mathbb{Z}$, then $n' = M - n$, $M \equiv 2q(c-1)$ and $g_{M-n} = 0$.] In this situation there is a singular vector v_n^s and possibly a singular vector $v_{n'}^s$, however, the latter (when existing) is a descendant of v_n^s . Thus, we have the following embedding diagram:

$$V^{h, c, q} \longrightarrow V^{h+|n|, c, q+\text{sign}(n)} \quad (3.122)$$

where the convention is used that the arrow points to the embedded module. This embedding has a kernel, since there is an infinite chain of embeddings of Verma modules:

$$\cdots \longrightarrow V_t \longrightarrow V_{t+1} \longrightarrow \cdots \quad (3.123)$$

where $V_t \equiv V^{h+t|n|, c, q+t \text{sign}(n)}$, $t \in \mathbb{Z}$. Using the Grassmannian properties of the odd generators one can show that this chain of embedding maps is exact. Due to the kernel

one has

$$\begin{aligned} \text{ch } L^{h,c,q} &= \text{ch } V^{h,c,q} - \widetilde{\text{ch}}_n V^{h+|n|,c,q+\text{sign}(n)} \\ &= \widetilde{\text{ch}}_n V^{h,c,q}. \end{aligned} \quad (3.124)$$

□

Proposition 3 ([158]). *The character formulae for the unitary cases A_2, P_2 , with $f_{1,2} = 0$ is given by*

$$\text{ch } L^{h,c,q} = \frac{(1-x)}{(1+x^{|n|}y^{\text{sign}(n)})(1+x^{|n|+1}y^{\text{sign}(n)})} \text{ch } V^{h,c,q} \quad (3.125)$$

Proof. The character relevant structure of $V^{h,c,q}$ is given by the embedding diagram:

$$\begin{array}{ccc} & V_0 = V^{h,c,q} & \\ \swarrow \text{---} & & \searrow \text{---} \\ V'_0 = V^{h+1,c,q} & & V_1 = V^{h+|n|,c,q+\text{sign}(n)} \\ \searrow \text{---} & & \swarrow \text{---} \\ & V'_1 = V^{h+|n|+2,c,q+\text{sign}(n)} & \end{array} \quad (3.126)$$

where the dashed arrows denote even embeddings, V_0 is reducible v. r. t. $g_n = 0 = f_{1,2}$ from the statement; then (with $\mu = 0$ for $\mathcal{A}$, $\mu = 1$ for $\mathcal{P}$)

$$h = \frac{1}{8}(c-1)(2n+\epsilon)^2 + n\epsilon + \frac{1}{8}\mu, \quad q = \frac{1}{2}(c-1)(2n+\epsilon), \quad \epsilon \equiv \text{sign}(n). \quad (3.127)$$

The other reducibilities relevant for the structure are V_1 w. r. t. $g_n = 0 = f_{1,2}$, V'_0 and V'_1 w. r. t. $g_{n+\text{sign}(n)} = 0$. Thus for the character formula follows:

$$\begin{aligned} \text{ch } L^{h,c,q} &= \text{ch } V^{h,c,q} - \text{ch } V^{h+1,c,q} - \widetilde{\text{ch}}_n V^{h+|n|,c,q+\text{sign}(n)} \\ &\quad + \widetilde{\text{ch}}_{n+\text{sign}(n)} V^{h+|n|+2,c,q+\text{sign}(n)}, \end{aligned} \quad (3.128)$$

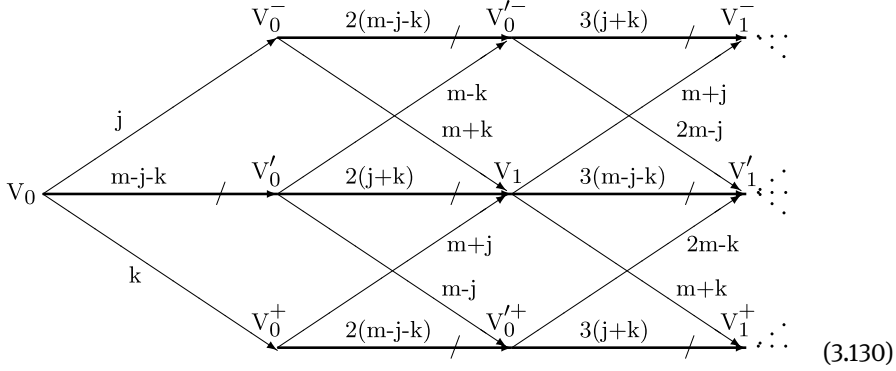
which after substituting the definitions gives (3.125). □

Proposition 4 ([158, 444, 385]). *The character formulae for the unitary cases $A_0, P_0^\pm$ is given by*

$$\begin{aligned} \text{ch } L_{m,j,k}(x, y) &= \sum_{n \in \mathbb{Z}_+} x^{mn^2+(j+k)n} \left\{ 1 - x^{(m-j-k)(2n+1)} \right. \\ &\quad + x^{mn+k} y \left[\frac{x^{2(m-j-k)(n+1)}}{1+x^{mn+m-j}y} - \frac{1}{1+x^{mn+k}y} \right] \\ &\quad + x^{mn+k} y^{-1} \left[\frac{x^{2(m-j-k)(n+1)}}{1+x^{mn+m-k}y^{-1}} - \frac{1}{1+x^{mn+j}y^{-1}} \right] \Big\} \\ &\quad \times \text{ch } V_{m,j,k}(x, y) \end{aligned} \quad (3.129)$$

where $L_{m,j,k} = L^{h,c,q}$, $V_{m,j,k} = V^{h,c,q}$, when h, c, q are expressed through m, j, k as in (3.113c) and (3.114c).

Proof. The structure of $V_0 \equiv V_{m,j,k}$ is given by the following embedding diagram:



$$V_n = V^{h+mn^2+(k+j)n,c,q}, \quad (3.131)$$

$$V'_n = V^{h+mn^2+m(2n+1)-(k+j)(n+1),c,q},$$

$$V_n^+ = V^{h+mn^2+(m+k+j)n+k,c,q+1},$$

$$V_n'^+ = V^{h+mn^2+m(3n+2)-(k+j)(n+2)+k,c,q+1},$$

$$V_n^- = V^{h+mn^2+(m+k+j)n+j,c,q-1},$$

$$V_n'^- = V^{h+mn^2+m(3n+2)-(k+j)(n+2)+j,c,q-1}.$$

From this it follows that

$$\begin{aligned} \text{ch } L_{m,j,k}(x, y) = \sum_{n \in \mathbb{Z}_+} [\text{ch } V_n - \text{ch } V'_n - \widetilde{\text{ch}}_{mn+k} V_n^+ - \widetilde{\text{ch}}_{mn+j} V_n^- \\ + \widetilde{\text{ch}}_{mn+m-j} V_n'^+ + \widetilde{\text{ch}}_{mn+m-k} V_n'^-], \end{aligned} \quad (3.132)$$

which after substituting the definitions gives (3.129). $\square$

Remark 1. It should be stressed that diagrams (3.126) and (3.130) represent the structure of the Verma modules V_0 , $V_{m,j,k}$, respectively, and the embeddings between the Verma modules in these diagrams. The structure as shown determines the character formulae (3.125) and (3.129) completely. Later it was shown that in (3.130) each even embedding between the Verma modules V_n and V'_n , $n = 1, 2, \dots$, and between the Verma modules V'_n and V_{n+1} , $n = 1, 2, \dots$, is generated by two uncharged fermionic singular vectors [191]. However, this refinement of embeddings has no relevance for the character formulae (3.129). $\diamond$

Proposition 5 ([158, 444, 385]). Let $V_{r,s}$, $r \in \mathbb{N}$, $s \in \mathbb{N} - 1/2$, be the Verma module $V^{h,c}$ with $h = h_{r,s}^T = [(tr - ms)^2 - t^2]/4mt + 1/8 = h_{m-r,t-s}^T$, $c = 1 - 2t/m$, $t, m \in \mathbb{N}$, $tr \leq ms$, $s < t < m$, t, m have no common divisor. Then the character formula for the corresponding irreducible quotient $L_{r,s}$ is given by

$$\text{ch } L_{r,s}(x) = \text{ch } V_{r,s}(x) \sum_{j \in \mathbb{Z}} x^{j(tm+j-tr-ms)} (1 - x^{s(2mj+r)}). \quad (3.133)$$

In particular, the character formula for the T_0 unitary cases $r \leq m/2$ is obtained from (3.133) by setting $t = 1$, $s = 1/2$.

The proof relies on the realization that the Verma modules $V_{r,s}$ has exactly the structure of certain Virasoro and $N = 1$ super-Virasoro (Neveu–Schwarz and Ramond) Verma modules for which the character formulae are known; cf. Section 3.4. $\diamond$

3.8 Modular invariants for theta-functions with characteristics and the twisted $N = 2$ superconformal and $\text{su}(2)$ Kac–Moody algebras

This section is based on [181]. We classify the modular invariants for Θ -functions with characteristics relevant for the twisted $N = 2$ super-Virasoro and the twisted $\text{su}(2)$ Kac–Moody algebras. Detailed and complete proofs are given. Based on this we give the classification of the modular invariants for the above algebras.

3.8.1 Preliminaries

The construction of the modular invariant partition functions on the torus is a crucial step towards the classification of two-dimensional conformal field theories [101, 336, 270, 98]. The classification of modular invariants for the Virasoro and $\text{su}(2)$ Kac–Moody algebras was conjectured in Cappelli–Itzykson–Zuber [98] and proved in [269, 268, 99, 367]; for the $N = 1$ super-Virasoro algebras was given in [97] and for the untwisted $N = 2$ super-Virasoro algebras in [268]. For the *twisted* $N = 2$ super-Virasoro algebras the study of modular invariance was done in [180–182].

We consider the discrete series of unitarizable highest weight modules (HWM) with central charge $c < 1$ (Subsection 3.8.2). We classify the modular invariants for the Θ -functions with characteristics (in terms of which are expressed the characters of the HWM) in a complete and detailed way (Subsection 3.8.3 and Appendix). Based on this we give the classification of the modular invariant partition functions for the twisted $n = 2$ superconformal algebra (or equivalently for the twisted $\text{su}(2)$ Kac–Moody algebra) (Subsection 3.8.4).

3.8.2 Characters of the twisted $N = 2$ superconformal and $\text{su}(2)$ Kac–Moody algebras

The main object of interest here are the following Θ -functions with characteristics $\varepsilon, \omega \in \{0, 1/2\}$:

$$\Theta_{r,k} \left[\begin{smallmatrix} \varepsilon \\ \omega \end{smallmatrix} \right] (\tau, z) = \sum_{j \in \mathbb{Z}} e^{2\pi i k (\tau(j+r/2k)^2 - z(j+r/2k))}, \quad (3.134)$$

where $k \in \frac{1}{2}\mathbb{N}$, $r \in \mathbb{Z} + \varepsilon \pmod{2k}$, $z, \tau \in \mathbb{C}$, $\text{Im } \tau > 0$. When $\varepsilon = \omega = 0$ and $k \in \mathbb{N}$ they coincide with the classical Θ -functions describing level k representations of $\text{su}(2)$ Kac–Moody algebras [354]. For $k = 1/2$ and $r = \varepsilon$ we obtain the classical Jacobi Θ -functions [460]:

$$\Theta_2 = \Theta\left[\begin{smallmatrix} 1/2 \\ 0 \end{smallmatrix}\right], \quad \Theta_3 = \Theta\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right], \quad \Theta_4 = \Theta\left[\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix}\right], \quad (3.135)$$

which are the even ones, and the odd one:

$$\Theta_1 = \Theta\left[\begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix}\right], \quad (3.136)$$

where we have denoted $\Theta\left[\begin{smallmatrix} \varepsilon \\ \omega \end{smallmatrix}\right] \equiv \Theta_{\varepsilon,1/2}\left[\begin{smallmatrix} \varepsilon \\ \omega \end{smallmatrix}\right]$.

Let $\mathcal{H}\left[\begin{smallmatrix} \varepsilon \\ \omega \end{smallmatrix}\right]$ be the $2k$ -dimensional space spanned by $\chi_r\left[\begin{smallmatrix} \varepsilon \\ \omega \end{smallmatrix}\right](\tau, z) = \Theta_{r,k}\left[\begin{smallmatrix} \varepsilon \\ \omega \end{smallmatrix}\right](\tau, z)/\eta(\tau)$ where $\eta(\tau)$ is the Dedekind η -function. The action of the modular group $\Gamma = \text{SL}_2(\mathbb{Z})$ on (z, τ) is given by [354]

$$\Gamma \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (z, \tau) = \left(\frac{z}{c\tau + d}, \frac{a\tau + b}{c\tau + d} \right). \quad (3.137)$$

One can take as generators of the modular group the matrices $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, thus $S : (\tau, z) \mapsto (-1/\tau, z/\tau)$ and $T : (\tau, z) \mapsto (\tau + 1, z)$. These generators act on the direct sum $\mathcal{H}$ of $\mathcal{H}\left[\begin{smallmatrix} \varepsilon \\ \omega \end{smallmatrix}\right]$ as follows [180]:

$$S : \left[\begin{smallmatrix} \varepsilon \\ \omega \end{smallmatrix}\right] \longrightarrow \left[\begin{smallmatrix} \omega \\ \varepsilon \end{smallmatrix}\right] \quad \text{with matrix elements} \quad S_{rr'} = \frac{C_S}{\sqrt{N}} e^{-2\pi i r r' / N}, \quad (3.138)$$

$$T : \left[\begin{smallmatrix} \varepsilon \\ \omega \end{smallmatrix}\right] \longrightarrow \left[\begin{smallmatrix} \varepsilon \\ \omega + \varepsilon + k \end{smallmatrix}\right] \quad \text{with matrix elements} \quad T_{rr'} = C_T e^{\pi i r^2 / N} \delta_{rr'}, \quad (3.139)$$

where C_S, C_T are phases that will cancel between the left and right chiral parts when we consider modular invariants. More explicitly we have

$$T \circ \left[\begin{smallmatrix} 1/2 \\ 0 \end{smallmatrix}\right] \xrightarrow{S} \left[\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix}\right] \xrightarrow{T} \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right] \circ S, \quad k \in \mathbb{N} - 1/2, \quad (3.140a)$$

$$T \circ \left[\begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix}\right] \circ S, \quad k \in \mathbb{N} - 1/2, \quad (3.140b)$$

$$T \circ \left[\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix}\right] \xrightarrow{S} \left[\begin{smallmatrix} 1/2 \\ 0 \end{smallmatrix}\right] \xrightarrow{T} \left[\begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix}\right] \circ S, \quad k \in \mathbb{N}, \quad (3.141a)$$

$$T \circ \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right] \circ S, \quad k \in \mathbb{N}. \quad (3.141b)$$

The characters of the $c < 1$ unitarizable HWM of the $n = 2$ superconformal algebras (SCA) were found in [158, 444]. (For the cases $c \geq 1$ and for some nonunitarizable HWM see [158, 162].) Later, it was pointed out that these characters coincide with the characters of the integrable HWM of the twisted $\text{su}(2)$ Kac–Moody algebra [512]. In [180] these characters were rewritten in terms of (3.134). The unitarizable representations of the SCA with $c < 1$ are parametrized by $c = 1 - 2/m$, $m \in \mathbb{N} + 1$. In the twisted case

the character corresponding to a field of conformal weight $h_r = c/24 + (m - 2r)^2/16m$, $r \in \mathbb{N}$, $1 \leq r \leq m/2$ is for m even:

$$\chi_r^T(\tau) = \Theta_{(m-2r)/4, m/4} \left[\begin{smallmatrix} \varepsilon \\ 1/2 \end{smallmatrix} \right] (\tau, z) / \Theta_4(\tau, z)|_{z=0}, \quad (3.142)$$

where

$$\varepsilon = \begin{cases} 0 & \text{if } m - 2r = 0 \bmod 4, \\ 1/2 & \text{if } m - 2r = 2 \bmod 4, \end{cases}$$

and for m odd:

$$\chi_r^T(\tau) = (\Theta_{(m-1)/2-r, m} \left[\begin{smallmatrix} 1/2 \\ 0 \end{smallmatrix} \right] (\tau, z) - \Theta_{(m+3)/2+r, m} \left[\begin{smallmatrix} 1/2 \\ 0 \end{smallmatrix} \right] (\tau, z)) / \Theta_4(\tau, z)|_{z=0}. \quad (3.143)$$

Let us introduce the following notation when $m = 2 \bmod 4$:

$$\Theta_s^{(0)} \left[\begin{smallmatrix} \varepsilon \\ \omega \end{smallmatrix} \right] \equiv \frac{\Theta_{s, m/4} \left[\begin{smallmatrix} \varepsilon \\ \omega \end{smallmatrix} \right]}{\Theta \left[\begin{smallmatrix} \varepsilon \\ \omega \end{smallmatrix} \right]}, \quad \left[\begin{smallmatrix} \varepsilon \\ \omega \end{smallmatrix} \right] \neq \left[\begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix} \right], \quad (3.144a)$$

$$\Theta_s^{(1/2)} \left[\begin{smallmatrix} \varepsilon \\ \omega \end{smallmatrix} \right] \equiv \frac{\Theta_{s, m/4} \left[\begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix} \right]}{\Theta \left[\begin{smallmatrix} \varepsilon \\ \omega \end{smallmatrix} \right]}, \quad \left[\begin{smallmatrix} \varepsilon \\ \omega \end{smallmatrix} \right] \neq \left[\begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix} \right], \quad (3.144b)$$

thus $\chi_r^T(\tau) = \Theta_{(m-2r)/4}^{(\varepsilon)} \left[\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix} \right]$ (with ε from (3.142)); and when $m = 0 \bmod 4$:

$$\Theta_s \left[\begin{smallmatrix} \varepsilon \\ \omega \end{smallmatrix} \right] \left[\begin{smallmatrix} \varepsilon' \\ \omega' \end{smallmatrix} \right] \equiv \frac{\Theta_{s, m/4} \left[\begin{smallmatrix} \varepsilon \\ \omega \end{smallmatrix} \right]}{\Theta \left[\begin{smallmatrix} \varepsilon' \\ \omega' \end{smallmatrix} \right]}, \quad \left[\begin{smallmatrix} \varepsilon \\ \omega \end{smallmatrix} \right] \neq \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right], \quad \left[\begin{smallmatrix} \varepsilon' \\ \omega' \end{smallmatrix} \right] \neq \left[\begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix} \right], \quad (3.145)$$

thus $\chi_r^T(\tau) = \Theta_{(m-2r)/4} \left[\begin{smallmatrix} \varepsilon \\ 1/2 \end{smallmatrix} \right] \left[\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix} \right]$. From (3.140) and (3.141) we obtain for $m = 2 \bmod 4$:

$$T \circ \Theta^{(\eta)} \left[\begin{smallmatrix} 1/2 \\ 0 \end{smallmatrix} \right] \xrightarrow{S} \Theta^{(\eta)} \left[\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix} \right] \xrightarrow{T} \Theta^{(\eta)} \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right] \circ S, \quad \eta = 0, 1/2, \quad (3.146)$$

and for $m = 0 \bmod 4$:

$$\begin{aligned} \Theta \left[\begin{smallmatrix} 1/2 \\ 0 \end{smallmatrix} \right] \left[\begin{smallmatrix} 1/2 \\ 0 \end{smallmatrix} \right] &\xrightarrow{S} \Theta \left[\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix} \right] \left[\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix} \right] \xrightarrow{T} \Theta \left[\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix} \right] \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right] \\ T \updownarrow & \qquad \qquad \qquad \updownarrow S \\ \Theta \left[\begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix} \right] \left[\begin{smallmatrix} 1/2 \\ 0 \end{smallmatrix} \right] &\xrightarrow{S} \Theta \left[\begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix} \right] \left[\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix} \right] \xrightarrow{T} \Theta \left[\begin{smallmatrix} 1/2 \\ 0 \end{smallmatrix} \right] \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right] \end{aligned} \quad (3.147)$$

Let Z_{st} , $s, t \in \{0, 1\}$, be the partition function on the torus where s (respectively, t) denote the boundary conditions in the space (respectively, time) direction so that the “0” (respectively, “1”) boundary condition is periodic (respectively antiperiodic). The partition function Z_{00} is a quadratic form of the untwisted characters, while Z_{10} is a quadratic form of the twisted characters (3.142) and (3.143). For $m = 2 \bmod 4$ Z_{01} (respectively, Z_{11}) is given in terms of $\Theta^{(\varepsilon)} \left[\begin{smallmatrix} 1/2 \\ 0 \end{smallmatrix} \right]$ (respectively, $\Theta^{(\varepsilon)} \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right]$), $\varepsilon = 0, 1/2$.

For $m = 0 \bmod 4$ Z_{01} (respectively, Z_{11}) is given in terms of $\Theta[\frac{1/2}{\varepsilon}][\frac{1/2}{0}]$, $\varepsilon = 0, 1/2$ (respectively $\Theta[\frac{0}{1/2}][\frac{0}{0}]$ and $\Theta[\frac{1/2}{0}][\frac{0}{0}]$). The $\mathbb{Z}_2$ orbifold partition function is given by

$$Z_{\text{orb}} = \frac{1}{2}(Z_{00} + Z_{\text{twist}}), \quad Z_{\text{twist}} = Z_{01} + Z_{10} + Z_{11}. \quad (3.148)$$

Here Z_{00} and Z_{twist} are modular invariant by themselves. Let $\mathcal{H}^{(s)}$ be the Hilbert space corresponding to $s = 0, 1$ boundary conditions. The combinations

$$\frac{1}{2}(Z_{s0} + Z_{s1}) \quad (3.149)$$

appearing in (3.148) should be interpreted as projections of a given parity on the Hilbert space $\mathcal{H}^{(s)}$.

We shall not consider the case m odd independently, since, as we saw in [180], the modular invariants for m odd are obtained from those with m' even if $m' = 4m$ (this is expected comparing (3.143) with (3.142) for $m = 0 \bmod 4$).

3.8.3 Modular invariants for Θ -functions with characteristics

The modular invariant partition functions are built as sesquilinear combinations of the characters which are expressed in terms of the Θ -functions with characteristics. In this section we shall obtain a classification of the modular invariants built from these Θ -functions and then from these we shall obtain the partition functions in Subsection 3.8.4.

Expanding the approach of [99, 269] we introduce a Heisenberg algebra of matrices $P[\frac{\varepsilon}{\omega}]$ and $Q[\frac{\varepsilon}{\omega}]$ acting on the space spanned by $\chi_r[\frac{\varepsilon}{\omega}] = \Theta_{r, N/2}[\frac{\varepsilon}{\omega}]/\eta$, $N \in \mathbb{N}$, $r \in \mathbb{Z} + \varepsilon \bmod N$, as follows (we skip denoting explicitly the sectors $[\frac{\varepsilon}{\omega}]$ when no confusion may arise):

$$Q\chi_r = e^{2\pi i r/N} \chi_r, \quad P\chi_r = \chi_{r-1}. \quad (3.150)$$

It is easy to see that the Weyl commutation relations hold: $QP = e^{2\pi i/N} PQ$, and the normalizations are

$$(P[\frac{\varepsilon}{\omega}])^N = e^{2\pi i \omega} 1[\frac{\varepsilon}{\omega}], \quad (Q[\frac{\varepsilon}{\omega}])^N = e^{2\pi i \varepsilon} 1[\frac{\varepsilon}{\omega}]. \quad (3.151)$$

The action of $\text{SL}_2(\mathbb{Z}_{2N})$ on the Θ -functions induces in an obvious way an action on the P and the Q . The set $\{P^k Q^\ell : k, \ell \bmod N\}$ spans the matrix algebra in the respective sector, thus averaging $P^k Q^\ell$ over $\text{SL}_2(\mathbb{Z}_{2N})$ for different k, ℓ will give all modular invariants we are looking for. We introduce the symbols

$$[k, \ell][\frac{\varepsilon}{\omega}] \equiv e^{\pi i k \ell / N} (P[\frac{\varepsilon}{\omega}])^k (Q[\frac{\varepsilon}{\omega}])^\ell, \quad (3.152)$$

where k, ℓ are mod $2N$ but are independent only mod N due to the normalization (3.151)

$$[k + aN, \ell + bN][\frac{\varepsilon}{\omega}] = e^{\pi i (a(\ell + 2\omega) + b(k + 2\varepsilon) + abN)} [k, \ell][\frac{\varepsilon}{\omega}]. \quad (3.153)$$

The generators of the modular group $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $T = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ act on these symbols as follows:

$$S^+[k, \ell] \begin{bmatrix} \varepsilon \\ \omega \end{bmatrix} S = [\ell, -k] \begin{bmatrix} \omega \\ \varepsilon \end{bmatrix} = ([k, \ell]S) \otimes ([\varepsilon, \omega]S), \quad (3.154)$$

$$T^+[k, \ell] \begin{bmatrix} \varepsilon \\ \omega \end{bmatrix} T = [k, \ell - k] \begin{bmatrix} \varepsilon \\ \omega + \varepsilon + N/2 \end{bmatrix} = ([k, \ell]T) \otimes ([\varepsilon, \omega]T). \quad (3.155)$$

Thus the action of a general element of $A \in \text{SL}_2(\mathbb{Z}_{2N})$ is realized as the action of A on the right of $[k, \ell]$ and the action of $A \bmod 2 \in \text{SL}_2(\mathbb{Z}_2)$ on $[\varepsilon, \omega]$ given by (3.140) and (3.141) with $k = N/2$.

Consider first the case N odd. Then the group of modular transformations factorizes $\text{SL}_2(\mathbb{Z}_{2N}) = \text{SL}_2(\mathbb{Z}_N) \times \text{SL}_2(\mathbb{Z}_2)$. $\text{SL}_2(\mathbb{Z}_2)$ is embedded into $\text{SL}_2(\mathbb{Z}_{2N})$ by

$$S = \begin{pmatrix} 1+N & N \\ N & 1+N \end{pmatrix}, \quad T = \begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix}, \quad (3.156)$$

and it permutes the sectors as in (3.140). Since from the $k, \ell \pmod{2N}$ symbols only $k, \ell \bmod N$ are independent we can restrict ourselves to $[2k, 2\ell]$ on which $\text{SL}_2(\mathbb{Z}_2)$, given by (3.156), acts trivially. Thus we obtain two sets of modular invariants:

$$\mathcal{M}_{k,\ell}^{(0)} = \frac{1}{C} \left(\sum [2k, 2\ell]A \right) \otimes \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/2 \end{bmatrix} + \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} \right) \quad (3.157)$$

and

$$\mathcal{M}_{k,\ell}^{(1/2)} = \frac{1}{C} \left(\sum [2k, 2\ell]A \right) \otimes \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}, \quad (3.158)$$

where $C = |\text{SL}_2(\mathbb{Z}_N)|$ and the sum is over $A \in \text{SL}_2(\mathbb{Z}_N) \subset \text{SL}_2(\mathbb{Z}_{2N})$. Obviously $\mathcal{M}_{k,\ell}^{(\eta)} A = \mathcal{M}_{k,\ell}^{(\eta)}$ for $A \in \text{SL}_2(\mathbb{Z}_{2N})$. Thus if $\alpha = \gcd(k, \ell)$ we can find a unimodular matrix A' with first column $(\ell/\alpha, -k/\alpha)$, hence $([2k, 2\ell] \begin{bmatrix} \varepsilon \\ \omega \end{bmatrix}) A' = [0, 2/\alpha] \begin{bmatrix} \varepsilon' \\ \omega' \end{bmatrix}$. Next we can find $A'' \in \text{SL}_2(\mathbb{Z}_2) \subset \text{SL}_2(\mathbb{Z}_{2N})$ such that $\begin{bmatrix} \varepsilon' \\ \omega' \end{bmatrix} A'' = \begin{bmatrix} \varepsilon \\ \omega \end{bmatrix}$; thus if $A = A' A''$ we get $\mathcal{M}_{k,\ell}^{(\eta)} = \mathcal{M}_{k,\ell}^{(\eta)} A = \mathcal{M}_{0,\alpha}^{(\eta)}$. However, $\mathcal{M}_{0,\alpha}^{(\eta)} = \mathcal{M}_{N,\alpha}^{(\eta)}$ and repeating the same argument with $\delta = \gcd(N, \alpha)$ we find that $\mathcal{M}_{k,\ell}^{(\eta)} = \mathcal{M}_{0,\delta}^{(\eta)}$. Thus when N is odd we have two sets of invariants $\mathcal{M}_{\delta}^{(\eta)} \equiv \mathcal{M}_{0,\delta}^{(\eta)}$ (cf. (3.157) and (3.158)) labelled by δ , the divisors of N .

Let us consider now N even. In this case $\text{SL}_2(\mathbb{Z}_{2N})$ does not factorize in a simple way and the analysis is more involved. (For details we refer to the appendix.) The action of the modular group on the sectors factorizes through $\mu_2^{2N} : \text{SL}_2(\mathbb{Z}_{2N}) \rightarrow \text{SL}_2(\mathbb{Z}_2)$ and is given by (3.141). For N even we shall not discuss the sector $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (which transforms by itself) since this case was considered in [99, 269] and moreover the twisted characters are expressed through $\begin{bmatrix} * \\ 1/2 \end{bmatrix}$.

The kernel Λ_N^{2N} of the map $\mu_N^{2N} : \text{SL}_2(\mathbb{Z}_{2N}) \rightarrow \text{SL}_2(\mathbb{Z}_N)$ is isomorphic to $K \equiv (\mathbb{Z}_2)^3$. An element $A \in \Lambda_N^{2N}$ has the form

$$A = \begin{pmatrix} 1 + \alpha N & \beta N \\ \gamma N & 1 + \alpha N \end{pmatrix}, \quad \alpha, \beta, \gamma = 0, 1. \quad (3.159)$$

The maximal subgroup that does not lead out of a sector is $\Lambda_N^{2N} \subset \Lambda_2^{2N}$. Using (3.159) and (3.153) we get

$$[k, \ell]A = e^{\pi i(a(2\varepsilon\ell+2k\omega)+\beta(k+2\varepsilon)+\gamma\ell(\ell+2\omega))} [k, \ell]. \quad (3.160)$$

Averaging over $K \subset \mathrm{SL}_2(\mathbb{Z}_{2N})$ we get

$$\langle [k, \ell] \begin{bmatrix} \varepsilon \\ \omega \end{bmatrix} \rangle_K = \frac{1}{2} (1 + e^{2\pi i(k\omega+\ell\varepsilon)}) [k, \ell] \begin{bmatrix} \varepsilon \\ \omega \end{bmatrix}. \quad (3.161)$$

Therefore we consider only symbols $[k, \ell]$ such that

$$\begin{aligned} k & \text{ is even in } \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}, \\ \ell & \text{ is even in } \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}, \\ k \text{ and } \ell & \text{ have the same parity in } \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}. \end{aligned} \quad (3.162)$$

Note also that from (3.153) we have in particular

$$\begin{aligned} [2k, \ell] &= [2k, 2\ell + N] \quad \text{in } \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}, \\ [k, 2\ell] &= [k + N, 2\ell + N] \quad \text{in } \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}, \\ [\ell, \ell + 2k] &= [\ell + N, \ell + 2k + N] \quad \text{in } \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}; \end{aligned} \quad (3.163)$$

thus in (3.162) we shall take $k, \ell \bmod N$. The modular invariants will be given by

$$\mathcal{M}_{k\ell} = \frac{1}{|\Lambda_2^{2N}|} \sum_{A \in \Lambda_2^{2N}} ([2k, \ell] \begin{bmatrix} 0 \\ 1/2 \end{bmatrix} + [\ell, -2k] \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} + [\ell, -\ell - 2k] \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}) A. \quad (3.164)$$

In what follows we shall concentrate on the sector $\begin{bmatrix} 0 \\ 1/2 \end{bmatrix}$. We want to find the independent invariants. Note that we can reduce the averaging over Λ_2^{2N} to one over Λ_2^N because K acts trivially on symbols of the form (3.162) and $\Lambda_2^N \cong \Lambda_2^{2N}/K$.

First let us consider symbols of the form [even, odd]. Let $\alpha = \gcd(2k, \ell)$ be odd and let ξ and η be Bézout multipliers such that $\eta\ell/\alpha + \xi 2k/\alpha = 1$, hence $[2k, \ell]A = [0, \alpha]$, $A = \begin{pmatrix} \ell/\alpha & \xi \\ -2k/\alpha & \eta \end{pmatrix}$. We want to have $A \in \Lambda_2^N$, i. e. to have odd entries on the diagonal and even entries off the diagonal. Obviously η is odd, while we may assume that ξ is even, because if it is not we shall take $\xi + \ell/\alpha$ (even), $\eta - 2k/\alpha$ (odd) as Bézout multipliers. Since $A \in \Lambda_2^N$ the symbols $[2k, \ell]$ and $[0, \alpha]$ lie on the same orbit of Λ_2^N and give the same invariant upon averaging. By the same reasoning $[0, \alpha] = [2N, \alpha]$ and $[0, \alpha], \delta = \gcd(2N, \alpha)$ odd, give the same invariant $\mathcal{M}_\delta = \mathcal{M}_{0\delta}$ from (3.164).

Next let us consider symbols of the form [even, even]. As we said, K leaves invariant all (3.162), thus averaging over Λ_2^{2N} is equivalent to averaging over $\Lambda_2^N \cong \Lambda_2^{2N}/K$. Let $N = 2^r \nu$ with ν odd. If $r \geq 2$ then $\Lambda_{N/2}^N \cong K$ and we check the effect of averaging $[2k, 2\ell]$ over it. Recall that in the sector $\begin{bmatrix} 0 \\ 1/2 \end{bmatrix}$ we have $[* + N, 2\ell] = -[* , 2\ell]$. Taking an element of $\Lambda_{N/2}^N$ we have

$$[2k, 2\ell] \begin{pmatrix} 1 + \alpha N/2 & \beta N/2 \\ \gamma N/2 & 1 + \alpha N/2 \end{pmatrix} = e^{\pi i(\alpha k + \gamma \ell)/2} [2k, 2\ell]. \quad (3.165)$$

Thus averaging over $\Lambda_{N/2}^N$ leaves $[2k, 2\ell]$ invariant if k and ℓ are both even and gives zero otherwise. Next consider the action of $\Lambda_2^N/K = \Lambda_2^{N/2}$ on $[4k, 4\ell]$. We repeat the argument $r - 1$ times. Thus finally we are left with the action of $\Lambda_2^{2^r} = \text{SL}_2(\mathbb{Z}_v)$ on $[2^r k, 2^r \ell]$. Similarly to the case that N is odd we conclude that the independent modular invariants among the ones coming from [even, even] are labelled by the divisors δ of N such that $\bar{\delta} = N/\delta$ is odd.

Putting everything together we obtain the following result.

Theorem 2. *The modular invariants which can be built from the functions $\Theta_{r,N/2}[\frac{\varepsilon}{\omega}]$ are given as follows. (a) For N odd there are two sets of invariants $\mathcal{M}_\delta^{(0)}$ ($= \mathcal{M}_{0,\delta}^{(0)}$ from (3.157)) and $\mathcal{M}_\delta^{(1/2)}$ ($= \mathcal{M}_{0,\delta}^{(1/2)}$ from (3.158)), labelled by δ , the divisors of N . (b) For N even the modular invariants are given by $\mathcal{M}_\delta$ ($= \mathcal{M}_{0,\delta}$ from (3.164)) in the sectors $[\frac{0}{1/2}]$, $[\frac{1/2}{0}]$ and $[\frac{1/2}{1/2}]$, where δ is a divisor of N such that either δ or N/δ is odd; for the sector $[\frac{0}{0}]$ the invariants (found in [269]) are labelled by all divisors of $N/2$. $\diamond$*

For later convenience we will present a different set of modular invariants also labelled by divisors of N . We will use a modified Kronecker δ , which is ω -periodic in N ,

$$\delta_{r,s}^{(\omega)} = \begin{cases} 0 & \text{if } N|r-s, \\ 1 & \text{if } 2N|r-s, \\ (-1)^{2\omega} & \text{if } N|r-s \text{ and } 2N|r-s; \end{cases} \quad (3.166)$$

thus $\omega = 0$ means periodic, while $\omega = 1/2$ means antiperiodic. For $\delta|N$, δ odd, we define

$$(\Omega_\delta[\frac{\varepsilon}{\omega}])_{\bar{r}r} = \sum_{x \bmod \bar{\delta}} \sum_{y \bmod \delta} \delta_{\bar{r},y\bar{\delta}+(x+\varepsilon)\delta}^{(\omega)} \delta_{r,y\bar{\delta}-(x+\varepsilon)\delta}^{(\omega)}, \quad (3.167)$$

where $\bar{\delta} = N/\delta$, $\bar{r}, r \in \mathbb{Z} + \varepsilon \bmod N$. As is well known the invariants are labelled by divisors δ such that either δ or $\bar{\delta}$ is odd. (We can use (3.167) also for δ even and $\bar{\delta}$ odd adding ε to y and not to x .) These Ω_δ can be written in the form (for δ odd and $\bar{\delta} = N/\delta$)

$$\delta_{r,s}^{(\omega)} = \begin{cases} e^{2\pi i \omega \alpha} & \text{if } \bar{\delta}|\bar{r} + r \text{ and } \alpha \text{ is the parity of } (\bar{r} + r)/\bar{\delta}, \\ 0 & \text{otherwise.} \end{cases} \quad (3.168)$$

We can check directly that the Ω are modular invariants. Under the S -transformation we have

$$\begin{aligned} (S^+ \Omega_\delta[\frac{\varepsilon}{\omega}] S)_{\bar{r}r} &= \sum_{s, \bar{\beta} \bmod N} \frac{e^{2\pi i \bar{\beta}/N}}{\sqrt{N}} \frac{e^{-2\pi i r s/N}}{\sqrt{N}} (\Omega_\delta[\frac{\varepsilon}{\omega}])_{\bar{\beta}s} \\ &= (\Omega_\delta[\frac{\varepsilon}{\omega}])_{\bar{r}r}. \end{aligned} \quad (3.169)$$

To show (3.169) one needs

$$\sum_{y \bmod \delta} e^{2\pi i a y/\delta} = \begin{cases} 1 & \text{if } \delta|a, \\ 0 & \text{otherwise,} \end{cases} \quad (3.170)$$

where a and y are integers. As an example let us see how we get by the S transformation from the $\begin{bmatrix} 0 \\ 1/2 \end{bmatrix}$ sector where we have integer r and antiperiodicity in N to the sector $\begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$ with half-integer modes r and periodicity in N . In (3.169) we shall have terms of the form

$$\sum_{r \bmod N} \delta_{r,a}^{(1/2)} e^{2\pi i r s / N} = e^{2\pi i a s / N}, \quad r \in \mathbb{Z}, s \in \mathbb{Z} + 1/2,$$

and this is an identity between antiperiodic under $a \mapsto a + N$ functions (the RHS is antiperiodic: $\exp(2\pi i(a+N)s/N) = -\exp(2\pi i a s / N)$ since $s \in \mathbb{Z} + 1/2$). Conversely, going from $\begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} 0 \\ 1/2 \end{bmatrix}$ we have

$$\sum_{s \bmod N} \delta_{s,a+1/2}^{(0)} e^{2\pi i s r / N} = e^{2\pi i a r / N} e^{\pi i r / N}, \quad r \in \mathbb{Z}, s \in \mathbb{Z} + 1/2,$$

and the second factor on the RHS builds up to an antiperiodic δ -function in $\Omega\begin{bmatrix} 0 \\ 1/2 \end{bmatrix}$.

Let us see now what happens under the T -transformation:

$$(T^+ \Omega_\delta \begin{bmatrix} \varepsilon \\ \omega \end{bmatrix} T)_{\bar{r}r} = e^{2\pi i \frac{(\bar{r}-r)(\bar{\delta}-s)}{2\delta\bar{\delta}}} (\Omega_\delta \begin{bmatrix} \varepsilon \\ \omega \end{bmatrix})_{\bar{r}r}. \quad (3.171)$$

When N is odd both δ and $\bar{\delta}$ are odd. If $\varepsilon = 1/2$, then $\bar{r} \pm r$ have different parity, the two in the denominator is compensated and thus ω does not change. If $\varepsilon = 0$ both $\bar{r} \pm r$ have the same parity, thus ω is changed. Therefore when N is odd T interchanges $\begin{bmatrix} 0 \\ 1/2 \end{bmatrix}$ and $\begin{bmatrix} \varepsilon \\ \omega + \varepsilon + 1/2 \end{bmatrix}$, as we know. When N is even, one of the two divisors is even, the other odd. If $\varepsilon = 0$ both $\bar{r} \pm r$ have the same parity which is even because one of them is divisible by an even number, thus the other compensates the two in the denominator and ω does not change. If $\varepsilon = 1/2$ the prefactor becomes the parity of $(\bar{r} + r)/\delta$ (if $\bar{\delta}$ is even) and changes ω . Therefore, again as it should be, T interchanges $\begin{bmatrix} \varepsilon \\ \omega \end{bmatrix}$ and $\begin{bmatrix} \varepsilon \\ \omega + \varepsilon \end{bmatrix}$ when N is even.

We want to check that Ω_δ are a maximal set of independent invariants as the $\mathcal{M}_\delta$. Using the general formula

$$\mathcal{M} = \frac{1}{N} \sum_{k,\ell \bmod N} P^k Q^\ell \text{Tr}(\mathcal{M} Q^{-\ell} P^{-k}), \quad (3.172)$$

which is true in each sector $\begin{bmatrix} \varepsilon \\ \omega \end{bmatrix}$, we find that $\Omega_\delta \begin{bmatrix} \varepsilon \\ \omega \end{bmatrix}$ is expressed through $\mathcal{M}_{\delta'} \begin{bmatrix} \varepsilon \\ \omega \end{bmatrix}$, δ' running over the appropriate divisors by an invertible matrix, thus showing that the set of Ω_δ is equivalent to the set of $\mathcal{M}_\delta$.

3.8.4 Modular invariant partition functions

In the expressions (3.142) and (3.143) for the twisted characters the Θ -functions are taken for $z = 0$. These Θ -constants have the property $\Theta_{r,k} \begin{bmatrix} \varepsilon \\ \omega \end{bmatrix} = \Theta_{-r,k} \begin{bmatrix} \varepsilon \\ \omega \end{bmatrix}$, which corresponds to the invariance of the twisted characters and the conformal weights under

the change $r \mapsto m - r$. Therefore a Θ -function modular invariant $\mathcal{M}_{\bar{r},r}$ will give an invariant quadratic form for the twisted characters if it satisfies $\mathcal{M}_{\bar{r},r} = \mathcal{M}_{-\bar{r},r} = \mathcal{M}_{\bar{r},-r}$. Hence all the invariants will be given by

$$\mathcal{M}\left[\frac{\varepsilon}{\omega}\right] = \sum_{\substack{\delta|N \\ \delta \text{ odd}}} C_{\delta}(\Omega_{\delta} + \Omega_{\bar{\delta}})\left[\frac{\varepsilon}{\omega}\right].$$

The requirements that the coefficients in the quadratic forms Z_{00} and Z_{10} have the meaning of operator content (i. e. these coefficients must be nonnegative integers and the vacuum must come with coefficient 1) and that $\frac{1}{2}(Z_{00} + Z_{01})$ is interpreted as a projection on the Hilbert space $\mathcal{H}^{(0)}$ leave us with the following possibilities:

- 1) the diagonal invariant present for all m ;
- 2) the diagonal invariant of $\left[\frac{\varepsilon}{\omega}\right] \neq \left[\frac{1/2}{1/2}\right]$ for $m = 2 \bmod 4$;
- 3) $(\Omega_1 + \Omega_6 - \Omega_2 - \Omega_3)\left[\frac{\varepsilon}{\omega}\right]$ for $m = 12$;
- 4) $(\Omega_1 + \Omega_9 - \Omega_3)\left[\frac{\varepsilon}{\omega}\right]$ of $\left[\frac{\varepsilon}{\omega}\right] \neq \left[\frac{1/2}{1/2}\right]$ for $m = 18$;
- 5) $(\Omega_1 + \Omega_{15} - \Omega_5 - \Omega_3)\left[\frac{\varepsilon}{\omega}\right]$ of $\left[\frac{\varepsilon}{\omega}\right] \neq \left[\frac{1/2}{1/2}\right]$ for $m = 30$.

These modular invariant partition functions will be considered in more detail in the next section.

3.8.5 Appendix

Here we have collected several simple facts together with their proofs. Let us denote $\mathbb{Z}_m = \mathbb{Z}/n\mathbb{Z}$; $\mathbb{Z}_m^*$ is for the units in $\mathbb{Z}_m$ (which have a multiplicative inverse mod m); $|G|$ for the number of elements of G ; $p|N$ means p divides N ; $\Gamma = \text{SL}_2(\mathbb{Z})$ for the modular group; $\Gamma(N) = \{A \in \Gamma : A \equiv 1 \bmod N\}$ is the order N principal congruence subgroup; $\Gamma^N = \text{SL}_2(\mathbb{Z}_N)$; $\gcd(a, b)$ the greatest common divisor of a and b .

The cornerstone of all arguments is the following simple fact. If a and b are relatively prime then there exist Bézout multipliers $c, d \in \mathbb{Z}$ such that $ad - bc = 1$ and all such multipliers are $\{c + ra, d + rb : r \in \mathbb{Z}\}$. In other words, given a two-vector with integer relatively prime entries it can be complemented to a matrix in $\text{SL}_2(\mathbb{Z})$. The fact follows if we show that the sets $\{0, 1, \dots, a-1\}$ and $\{kb \bmod a : k = 0, 1, \dots, a-1\}$ coincide. Suppose not, then we must have $aj - i$ for some $0 \leq i < j \leq a-1$, which is a contradiction.

Next we consider the sequence

$$1 \rightarrow \Gamma(N) \rightarrow \Gamma \rightarrow \Gamma^N \rightarrow 1, \quad (3.173)$$

where the homomorphism $\Gamma \rightarrow \Gamma^N$ is realized by mod N . This sequence is exact (see e. g. [404]); so we may identify Γ^N with $\Gamma/\Gamma(N)$.

Next we want to prove

$$|\Gamma^N| = N^3 \prod_{p|N} (1 - 1/p^2), \quad (3.174)$$

where the product is over the prime divisors of N not counting multiplicities. Note that if p is prime then $\mathbb{Z}_p^* = \mathbb{Z}_p \setminus \{0\} = (\mathbb{Z} - p\mathbb{Z})/p\mathbb{Z}$; more generally $\mathbb{Z}_{p^r}^* = (\mathbb{Z} - p\mathbb{Z})/p^r\mathbb{Z}$, thus $|\mathbb{Z}_{p^r}^*| = p^r - p^{r-1}$. First assume $N = p$, p a prime. Take $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ so that at least one is a unit. There are $p^2 - 1$ such choices. Let $k = \gcd(a, b)$ (by convention $\gcd(0, b) = b$). This k is a unit so there is a k' such that $kk' = 1 \pmod{p}$. Let $\{c + r\alpha, d + r\beta : r \in \mathbb{Z}\}$ be the multipliers for $\alpha = a/k, \beta = b/k$. Then

$$\begin{pmatrix} a & b \\ k'(c + r\alpha) & k'(d + r\beta) \end{pmatrix} \in \Gamma^p,$$

where for $r \in \mathbb{Z}_p$ we have p choices. Thus $|\Gamma^p| = p(p^2 - 1)$. Second, assume $N = p^r$, p a prime, $r \in \mathbb{N}$. Take (a, b) as above. There are $p^{r-1}(p^r - p^{r-1}) + (p^r - p^{r-1})p^r = p^{2r} - p^{2(r-1)} = N^2(1 - 1/p^2)$ such choices. Completing $a/k, b/k$ as above to an element of Γ^N gives N more choices, thus $|\Gamma^N| = N^3(1 - 1/p^2)$. Third, let $N = pq$, with p, q primes. Again take a, b so at least one is a unit. There are $(p^2 - 1)(q^2 - 1)$ such choices (because the number of pairs such that both are zero divisors is $p^2 + q^2 - 1$). Proceeding as above we find the order of Γ^N to be $pq(p^2 - 1)(q^2 - 1) = N^3(1 - 1/p^2)(1 - 1/q^2)$. Combining the above gives the general case.

If $n|N$ let us denote by μ_n^N the canonical homomorphism $\Gamma^N \rightarrow \Gamma^n$ realized by mod n and by Λ_n^N its kernel. From the surjectivity of mod N in (3.173) we see that μ_n^N is surjective. If $N = mn$ and $\gcd(m, n) = 1$ then $|\Gamma^N| = |\Gamma^m||\Gamma^n|$ and $\Gamma^N = \Gamma^m \times \Gamma^n$; moreover, $\Lambda_p^N \cong \Gamma^{N/p}$ for $p = m, n$. If $L|M|N$ then $\mu_L^M = \mu_L^N \circ \mu_N^M$ and $\Lambda_N^M \subset \Lambda_L^M$. If N is even Λ_N^{2N} is isomorphic to $K \cong (\mathbb{Z}_2)^3$, consisting of the eight elements

$$\begin{pmatrix} 1 + \alpha N & \beta N \\ \gamma N & 1 + \alpha N \end{pmatrix}, \quad \alpha, \beta, \gamma = 0, 1, \quad (3.175)$$

and $\Lambda_N^{2N} \subset \Lambda_2^{2N}$. The group Γ^2 consists of six elements. It can be represented as generated by S and T with the relations $S^2 = T^2 = (ST)^3 = 1$. (Thus Γ^2 is isomorphic to the Weyl group of $\mathfrak{sl}(3, \mathbb{C})$.) By definition the sequence

$$1 \rightarrow \Lambda_2^{2N} \rightarrow \Gamma^{2N} \rightarrow \Gamma^2 \rightarrow 1$$

is exact. The group Γ^2 acts by permuting the sectors $[\frac{\varepsilon}{\omega}]$, while Λ_2^{2N} is their maximal invariance subgroup (it is generated by T^2 and ST^2S).

Let $N = 2^r v$ where n is odd. Obviously we have

$$\begin{array}{ccccccc} \Gamma^{2N} & \rightarrow & \Gamma^N & \rightarrow & \cdots & \rightarrow & \Gamma^{4n} & \rightarrow & \Gamma^{2n} & \rightarrow & \Gamma^n \\ \uparrow & \rightarrow & \uparrow & \rightarrow & \cdots & \rightarrow & \uparrow & \rightarrow & \uparrow & \rightarrow & \uparrow \\ K & \rightarrow & K & \rightarrow & \cdots & \rightarrow & K & \rightarrow & \Gamma^2 & \rightarrow & K \end{array} \quad (3.176)$$

where each $\uparrow \rightarrow$ is a short exact sequence. Next we obtain

$$\begin{array}{ccccccc} \Lambda_2^{2N} & \rightarrow & \Lambda_2^N & \rightarrow & \cdots & \rightarrow & \Lambda_2^{2n} & \rightarrow & \Gamma^n & \rightarrow & \Gamma^2 \\ \uparrow & \rightarrow & \uparrow & \rightarrow & \cdots & \rightarrow & \uparrow & \rightarrow & \uparrow & \rightarrow & \uparrow \\ K & \rightarrow & K & \rightarrow & \cdots & \rightarrow & K & \rightarrow & \Gamma^2 & \rightarrow & K \end{array}. \quad (3.177)$$

In the main text we use this diagram to “peel off” powers of two by averaging over K , which either gives zero or leaves things invariant.

3.9 Classification of modular invariant partition functions for the twisted $N = 2$ superconformal algebra, twisted $\text{su}(2)$ Kac–Moody algebra and D_{2k} parafermions

This section is based on [182]. The characters of the twisted $N = 2$ super-Virasoro algebra (or equivalently the twisted $\text{su}(2)$ Kac–Moody algebra) and the closely related characters for the twisted sector of the parafermion algebra are represented in terms of order N Θ constants with characteristics. All modular invariants for order N Θ -functions with characteristics are characterized by $\delta|N$ such that either δ or N/δ is odd. The modular invariant $\mathbb{Z}_2$ orbifold partition functions for theories with $c = 3k/(k+2)$ are found and classified in an *A-D-E series*.

3.9.1 Preliminaries

The $c = 3(1 - 2/m)$ representations of the $N = 2$ super-Virasoro algebra and the level $k(= m - 2)$ representations of the affine $\text{su}(2)$ algebra are closely related. Both can be decomposed as a product of a $\mathbb{Z}_k$ parafermionic current algebra [212] and a single boson. There is an integrable marginal operator that changes the radius of the boson, deforming the $N = 2$ case into $\text{su}(2)$ [580]. Thus a picture emerges, analogous to the $c = 1$ case (for the classification of $c = 1$ conformal theories, see, e. g. [274, 146, 90, 386]) which in particular contains a $\mathbb{Z}_2$ orbifold line containing the twisted $N = 2$ and $\text{su}(2)$ for special values of the compactification radius. For the orbifold partition function one has $Z_{\text{orb}}(R) = (Z(R) + Z_{\text{twist}})/2$, where $Z(R)$ is the untwisted partition function and Z_{twist} does not depend on the radius R . Based on the classification of modular invariants for Θ -functions with characteristics we classify all the modular invariant $\mathbb{Z}_2$ orbifold partition functions for theories with $c = 3k/(k+2)$. The results are contained in formulae (3.227)–(3.232); cf. [182].

The difference between the characters of the twisted $N = 2$ and the characters for the c -disorder sector of the parafermion algebra is irrelevant as far as modular transformations are concerned, thus the modular invariants will be the same (cf. (3.227)–(3.232)).

3.9.2 Parafermions, $\text{su}(2)$, $N = 2$ and the untwisted line of conformal models

The moduli space of $c = 1$ theories consists of two lines parametrized by the radius of the compactification circle and by the radius of the $S^1/\mathbb{Z}_2$ orbifold which lines intersect

at a multicritical point having a symmetry enhanced up to $\text{su}(2)$, level 1, and three disjoint points corresponding to $\text{su}(2)$ moded out by polyhedra subgroups. The motion along the lines is realized by marginal operators. At $R_{\text{circle}} = \sqrt{3}$ and, its dual $\sqrt{3}/2$ the symmetry is enhanced to $N = 2$ supersymmetry, while at $R_{\text{orbifold}} = \sqrt{3}$ and $\sqrt{3}/2$ one gets twisted $N = 2$ supersymmetry. This picture was put together by [274] and [146] and was proved to give a complete classification of $c = 1$ rational conformal theories in [386].

The algebra of parafermionic currents $\psi_\ell(z)$, $\ell = 0, 1, k-1$, $\psi_0 = I$, $\psi_\ell^\dagger = \psi_{k-\ell}$, of conformal weight $h_\ell^{pf} = \ell(k-\ell)/k$, was introduced by Zamolodchikov and Fateev [212] as a $\mathbb{Z}_k$ generalization of the $\mathbb{Z}_2$ Ising fermions:

$$\begin{aligned}\psi_{\ell_1}(z)\psi_{\ell_2}(0) &= c_{\ell_1\ell_2} z^{-2\ell_1\ell_2/k} \psi_{\ell_1+\ell_2}(0), \\ \psi_\ell(z)\psi_\ell^\dagger(0) &= z^{-2h_\ell} \left(I + \frac{2h_\ell}{c_{pf}} z^2 T^{pf}(0) + \dots \right),\end{aligned}\quad (3.178)$$

where T^{pf} is the parafermion stress energy tensor closing a Virasoro algebra with central charge $c_{pf} = 2(k-1)/(k+2)$.

Both the central charge $c = 3k/(k+2)$ $N = 2$ superconformal algebra and the level k $\text{SU}(2)$ Kac–Moody algebra can be described by tensoring (see [586] and [212] respectively) a $\mathbb{Z}_k$ parafermion with a free boson φ (however, with different compactification radius R for $N = 2$ and $\text{su}(2)$). It is easy to check that

$$\begin{aligned}J^\pm(z) &= \sqrt{k}\varphi_{\pm 1}(z) : \exp \frac{i}{\sqrt{k}}\varphi(z) : \quad (\psi_{-1} = \psi_{k-1}), \\ J^3(z) &= \sqrt{k}\varphi'(z),\end{aligned}\quad (3.179)$$

respectively,

$$\begin{aligned}G^\pm(z) &= \sqrt{2c/3}\varphi_{\pm 1}(z) : \exp \pm i\sqrt{(k+2)/2k}\varphi(z) :, \\ J(z) &= \sqrt{c/2}u\varphi'(z),\end{aligned}\quad (3.180)$$

together with the total stress energy $T(z) = T^{pf}(z) + T_{\text{boson}}(z)$ close the level k $\text{su}(2)$ current algebra with central charge $c = 1 + c^{pf} = 3k/(k+2)$, respectively the $N = 2$ superconformal algebra with the same central charge.

The characters of the level k representations of the $\text{su}(2)$ Kac–Moody algebra are [354]

$$\chi_{r,k}^{\text{aff}}(\tau, z) = \frac{\Theta_{r,k+2} - \Theta_{-r,k+2}}{\Theta_{1,2} - \Theta_{-1,2}}(\tau, z) = \sum_{s=-k+1}^k c_s^{r-1}(z) \Theta_{s,k}(\tau, z) \quad (3.181)$$

where $(r-1)/2$ is the spin, $\Theta_{r,u}(\tau, z) = \Theta_{r,u,z}(\tau, z, u=0)$

$$\Theta_{r,n}(\tau, z, u) = e^{-2\pi i n u} \sum_{j \in \mathbb{Z} + r/2n} e^{2\pi i n(j^2 \tau - jz)}, \quad r \in \mathbb{Z}_{2n}, \quad (3.182)$$

are order $2n$ Θ -functions, c_s^ℓ are the string functions. The investigation of modular invariance was carried out in [336, 270, 98, 269, 268] and culminated in the A-D-E classification [99, 367]:

$$Z_k^{\text{aff}}(\mathcal{G}) = \sum_{r, \bar{r}} N_{r, \bar{r}}(\mathcal{G}) \chi_{r, k}^{\text{aff}} \bar{\chi}_{\bar{r}, k}^{\text{aff}}, \quad (3.183)$$

$$\begin{aligned} N(A_{m-1}) &= \Omega_m, \quad m \in \mathbb{N} + 1, \\ N(D_{m/2} + 1) &= \Omega_m + \Omega_2, \quad m \in 2\mathbb{N} + 4, \\ N(E_6) &= \Omega_n + \Omega_j + \Omega_2, \quad m = 12, \\ N(E_7) &= \Omega_{18} + \Omega_3 + \Omega_2, \quad m = 18, \\ N(E_8) &= \Omega_{30} + \Omega_5 + \Omega_3 + \Omega_2, \quad m = 30, \end{aligned} \quad (3.184)$$

where $m = k + 2$ and

$$(\Omega_\delta)_{r\bar{r}} = \begin{cases} 1 & \text{if } 2\delta|\bar{r} - r \text{ and } 2\bar{\delta}|r + \bar{r}, \\ 0 & \text{otherwise,} \end{cases} \quad (3.185)$$

for $\delta|m$ and $\bar{\delta}\bar{m} = m$.

Using the connection (3.179) between parafermions and affine $\text{su}(2)$ and the character formula (3.181) Gepner and Qiu found that the characters for the $\mathbb{Z}_k$ parafermions are simply the level k $\text{su}(2)$ string functions times the Dedekind η -function [269]. For the $N = 2$ superconformal algebra, depending on the boundary conditions for the supercurrents $G^\pm = (G^1 \pm iG^2)\sqrt{2}$, one has NS when both $b^\pm$ are periodic, R when both are antiperiodic, and T (twisted) when G^1 (respectively G^2) is periodic, while G^2 (respectively G^1) is antiperiodic.

The characters of the $c < 1$ unitarizable representations of the $N = 2$ superconformal algebras were found in [158, 444]. Because of the factorization of the $N = 2$ theory into a product of parafermionic times a free boson, the $N = 2$ character can be put in a form similar to the last expression in (3.181), i. e. a sum of terms consisting of a string function (a parafermionic character) times a Θ -function (a free boson character). Of course, since the radius of the boson is different from the case of $\text{su}(2)$, so is the level of the Θ -functions. The modular invariants for the untwisted $N = 2$ are given by [268]

$$\begin{aligned} Z_{k, \pm}^{N=2}(\mathcal{G}) &= \frac{1}{4} \sum_{\ell, \bar{\ell}=0}^k \sum_{m=-k-1}^k N_{\ell\bar{\ell}}(\mathcal{G}) (\chi_{\ell, m}^{\text{NS}} \chi_{\bar{\ell}, m}^{\text{NS}*} + \chi_{\ell, m}^{\bar{\text{NS}}} \chi_{\bar{\ell}, m}^{\bar{\text{NS}}*} \\ &\quad + \chi_{\ell, m}^R \chi_{\bar{\ell}, m}^{R*} \pm \chi_{\ell, m}^{\tilde{R}} \chi_{\bar{\ell}, m}^{\tilde{R}*}) \end{aligned} \quad (3.186)$$

where $N_{\ell\bar{\ell}}(\mathcal{G})$ are given by (3.184) and the characters for $\bar{\text{NS}}$ and $\tilde{R}$ are obtained by inserting the fermion parity operator in the trace.

Having the example of the Gaussian line of critical theories in the case of $c = 1$, one would like to describe its analogue in the case $c = 3k/(k + 2)$. Yang has noticed

[580] that the partition function introduced in [144]

$$Z_k(\mathcal{G}, R) = \sum_{r, s \in \mathbb{Z}_k} \mathcal{Z}_k(r, s, \mathcal{G}) \hat{Z}_k(r, s, R), \quad (3.187)$$

where

$$\mathcal{Z}_k(r, s, \mathcal{G}) = \frac{1}{2} |\eta|^2 \sum_{\mu=-k+1}^k \sum_{\ell, \bar{\ell}=0}^k e^{2\pi i(\mu-r)s/k} N_{\ell\bar{\ell}}(\mathcal{G}) c_{\mu}^{\ell} c_{\mu-2r}^{\bar{\ell}*} \quad (3.188)$$

is the $\mathbb{Z}_k$ -twisted parafermionic partition function,

$$\hat{Z}_k(r, s, R) = \frac{1}{k|\eta|^2} \sum_{\mu=k+1}^k \sum_{m, n \in \mathbb{Z}} e^{2\pi i(r-\mu)s/k} q^{h_{mn}^+} \bar{q}^{h_{mn}^-}, \quad (3.189)$$

$$h_{mn}^{\pm} = \frac{1}{2} \left(\frac{1}{2R} \left(m + n - \frac{\mu - r}{k} \right) \mp R(km - kn + r) \right)^2, \quad (3.190)$$

is the $\mathbb{Z}_k$ twisted boson partition function, describes the modular invariant partition function for arbitrary R interpolating between the $\text{su}(2)$ partition function (3.183) for $R = 1/\sqrt{2}k$ and the $N = 2$ partition function (3.186) (with sign “−”) for $R = \frac{1}{2}\sqrt{\frac{k+2}{k}}$. The operator product algebra generated by a pair of oppositely charged currents of spin $s \in \frac{1}{2}\mathbb{N}$ is studied in [489]. The currents split in a product of a $U(1)$ -field and a parafermionic current. These theories are discrete points on the line parametrized by R .

3.9.3 Twisted sector of $N = 2$ and $\text{su}(2)$ and c -disorder fields

The $\mathbb{Z}_k$ symmetry of the parafermion model is enlarged to a D_k symmetry (the dihedral group of order $2k$) when complex conjugation C is included. To each element of D_k corresponds a disorder field. To the elements containing C correspond the so-called C -disorder fields [586]. When a parafermionic current is taken around a C -disorder field it goes into its conjugate. In the boson theory one may consider a twisted sector build on top of two twist fields of dimension $(1/16, 1/16)$. Taking the field φ around one of the twist fields gives $\varphi \rightarrow -\varphi$, while circling the other gives $\varphi \rightarrow \varphi + \pi R$. Using equation (3.180) for the $N = 2$ currents one sees that the twisted sector of $N = 2$ can be realized by combining c -disorder fields and bosonic twist fields [506, 500, 586].

The characters for the twisted $N = 2$ superconformal algebra with $c = 3k/(k+2)$, corresponding to a primary twisted field of conformal dimension $h_r = c/24 + (m - 2r)^2/16m$, $m = k + 2$ were found in [158, 444]:

$$\chi_r^T(\tau) = \frac{\Theta_{m/2-r, m} - \Theta_{m/2+r, m}}{\Theta_{0,2} - \Theta_{-2,2}} \Big|_{z=0}, \quad r \in \mathbb{N}, \quad 1 \leq r \leq m/2. \quad (3.191)$$

The modular properties of (3.191) will be considered in the next section. As in the untwisted case from the connection (3.180) between $N = 2$ and parafermions the characters corresponding to C -disorder fields can be obtained [506, 500]. The difference from (3.191) is only in the denominator which for parafermions becomes

$$D_t = \Theta_{-1/2,3} - \Theta_{-7/2,3} = q^{1/48} \prod_{n=1}^{\infty} (1 - q^{n/2}). \quad (3.192)$$

Under modular transformations one easily gets (cf. [500])

$$T \circ \tilde{D}_+ \xleftrightarrow{S} D_t \xleftrightarrow{T} \tilde{D}_+ \circ S$$

and this coincides with the modular properties of the denominator of (3.191) which is the classical Jacobi Θ -function Θ_4 (see next section). Therefore the invariants for the twisted sector of the $N = 2$ SCA and the C -disordered sector of the parafermionic algebra are the same.

In general if the operator algebra admits an automorphism $\alpha, \alpha^2 = 1$, we obtain a $\mathbb{Z}_2$ orbifold by projecting on invariant states. The untwisted partition function can be written as $Z = \sum_{\ell\bar{\ell}} N_{\ell\bar{\ell}} \text{tr}_{\ell\bar{\ell}}(q^{L_0-c/24} \bar{q}^{L_0-c/24})$ where $(\ell, \bar{\ell})$ label the set of primary operators in the theory, $N_{\ell\bar{\ell}} \in \mathbb{Z}_+$ (the modular invariant form) gives the operator content and the trace is taken over the tensor product of the left H_ℓ and right $H_{\bar{\ell}}$ Hilbert spaces. Moding out by $\mathbb{Z}_2$ means to insert a projector in the trace, i. e. to consider

$$\begin{aligned} Z' &= \sum_{\ell\bar{\ell}} N_{\ell\bar{\ell}} \text{tr}_{\ell\bar{\ell}} \left(\left(\frac{1 + \alpha\bar{\alpha}}{2} \right) q^{L_0-c/24} \bar{q}^{L_0-c/24} \right) \\ &= \frac{1}{2} (Z + Z_{PT}) \end{aligned} \quad (3.193)$$

where Z_{PT} means that on the torus we have periodic boundary conditions in the space direction (the trace is taken over the untwisted Hilbert space) and antiperiodic in the time direction (α is inserted in the trace). On the other hand (3.193) is not modular invariant, hence under S, T we have

$$T \circ Z_{PT} \xleftrightarrow{S} Z_{TP} \xleftrightarrow{T} Z_{TT} \circ S \quad (3.194)$$

where in Z_{T*} the traces are taken over the twisted Hilbert spaces (antiperiodic boundary conditions in the space direction). Thus the modular invariant orbifold partition function will be

$$Z_{\text{orb}} = \frac{1}{2} (Z + Z_{\text{twist}}), \quad Z_{\text{twist}} = Z_{PT} + Z_{PP} + Z_{TT}. \quad (3.195)$$

In the case of $c = 1$ Z_{twist} does not depend on the compactification radius R , because the marginal operator that changes the radius does not couple to the twist fields; thus Z_{orb} depends on R through Z .

The twisted $\mathfrak{su}(2)$ Kac–Moody algebra is obtained when two of the three generators $J^1(z), J^2(z), J^3(z)$ become antiperiodic as z circles the origin. Because this twisting is realized by an inner automorphism the twisted $\mathfrak{su}(2)$ is isomorphic to the untwisted $\mathfrak{su}(2)$ KM algebra. In spite of this the stress energy tensor obtained by the Sugawara formula is different in the twisted and untwisted case; hence the characters are different. The latter were calculated in [512] and coincide with the twisted $N = 2$ characters (3.191). This comes as no surprise in view of the fact that Z_{twist} in (3.195) does not depend on the radius, and the difference between $\mathfrak{su}(2)$ and $N = 2$ is only a difference of R .

Determining the orbifold modular invariants (3.195) will involve the following: finding the modular invariants in the twisted sector (3.194) and then imposing the requirements that the Z_{twist} invariants are consistent with the untwisted invariants (3.186) through (3.193) and that the modular invariant form for Z_{TP} (the form in which the twisted characters participate) has nonnegative entries. Because Z_{twist} does not depend on R we can make the connection (3.193) between the twisted and untwisted invariants for the case of $\mathfrak{su}(2)$. Thus if the twist automorphism is $\alpha : \alpha(J^3) = J^3, \alpha(J^i) = -J^i, i = 1, 2$, instead of the untwisted character (3.181) which are $\chi_r = \text{tr}_r(q^{L_0 - c/24} y^{J_0^3})$ where $y = \exp(-2\pi iz)$ and J_0^3 is the zero mode of $J^3(z)$, we have to consider

$$\tilde{\chi}_r = \text{tr}(\alpha q^{L_0 - c/24} y^{J_0^3})|_{z=0}.$$

The insertion of α is equivalent to changing $z \rightarrow z + 1/2$. Thus in (3.181) we get

$$\begin{aligned} (\Theta_{r,m} - \Theta_{-r,m})(z = 1/2) &= (1 - \exp(\pi i r)) \exp\left(\frac{-\pi i r}{2}\right) \Theta_{r,m}(z = 0) \\ &= \begin{cases} 2 \exp(\frac{-\pi i r}{2}) \Theta_{r,m} & \text{if } r \text{ odd,} \\ 0 & \text{if } r \text{ even,} \end{cases} \end{aligned}$$

and

$$\tilde{\chi}_r = \begin{cases} \exp(\pi i(1-r)/2) \Theta_{r,m} / \Theta_{1,2} & \text{if } r \text{ odd,} \\ 0 & \text{if } r \text{ even.} \end{cases} \quad (3.196)$$

The twisted characters χ_r^T are given by the trace of $q^{L_0 - c/24}$ over the Hilbert space $\mathcal{H}_r^T$ (having as a cyclic vector $\varphi_r^T(0)|0\rangle$, where φ_r^T is the twisted primary field of conformal dimension h_r). Inserting the twisting automorphism under the trace we obtain the functions $\tilde{\chi}_r^T$ out of which one constructs the quadratic form Z_{TT} . Thus $\frac{1}{2}(Z_{TP} + Z_{TT})$ results in a projection on a subspace of $\mathcal{H}^T$ with definite parity.

3.9.4 Modular invariants for the $\mathbb{Z}_2$ orbifold line

In this section we assume that $m = k + 2$ is even. In [180] in order to study the modular properties of the twisted characters (3.191) we have written them in terms of Θ

functions with characteristics

$$\Theta_{r,N/2}[\frac{\varepsilon}{\omega}](\tau, z) = \sum_{j \in \mathbb{Z}} \exp(2\pi i \omega) \exp 2\pi i \frac{N}{2} \left(\tau \left(j + \frac{r}{N} \right)^2 - z \left(j + \frac{r}{N} \right) \right) \quad (3.197)$$

where $r \in \mathbb{Z} + \varepsilon \pmod{N}$ and $\varepsilon, \omega = 0, 1/2 \pmod{1}$. This is a Θ -function of order N . When N is even and $\varepsilon = 0 = \omega$ this coincides with the Θ functions defined in (3.182). Let $\mathcal{H}[\frac{\varepsilon}{\omega}]$ be the N -dimensional space spanned by $\Theta_{r,N/2}[\frac{\varepsilon}{\omega}](\tau, z)/\eta(\tau)$, where η is the Dedekind function. The generators of the modular group Γ act as $S: (\tau, z) \mapsto (-1/\tau, z/\tau)$ and $T: (\tau, z) \mapsto (\tau+1, z)$. They act on the direct sum $\mathcal{H}$ of the different $\mathcal{H}[\frac{\varepsilon}{\omega}]$ as follows:

$$S: [\frac{\varepsilon}{\omega}] \longleftrightarrow [\frac{\omega}{\varepsilon}] \text{ with matrix elements } S_{rr'} = \frac{c_S}{\sqrt{N}} \exp(2\pi i r r' / N), \quad (3.198)$$

$$T: [\frac{\varepsilon}{\omega}] \longleftrightarrow [\frac{\varepsilon}{\omega + \varepsilon + N/2}] \text{ with matrix elements } T_{rr'} = c_T \exp(\pi i r^2 / N) \delta_{rr'} \quad (3.199)$$

where c_S, c_T are phases, irrelevant in what follows. The principal congruence subgroup $\Gamma(2) = \{A \in \Gamma : A \equiv \mathbb{1} \pmod{2}\}$ permutes the sectors as follows;

$$N_{\text{odd}} : T \circ [\frac{1/2}{0}] \xleftrightarrow{S} [\frac{0}{1/2}] \xleftrightarrow{T} [\frac{0}{0}] \circ S, \quad T \circ [\frac{1/2}{1/2}] \circ S, \quad (3.200)$$

$$N_{\text{even}} : T \circ [\frac{0}{1/2}] \xleftrightarrow{S} [\frac{1/2}{0}] \xleftrightarrow{T} [\frac{1/2}{1/2}] \circ S, \quad T \circ [\frac{0}{0}] \circ S. \quad (3.201)$$

In Section 3.8 (cf. [181]) we have proved that the modular invariants for the above Θ -functions (3.197) are labelled by the divisors δ of N and that either δ or $\bar{\delta} = N/\delta$ is odd and are

$$(\Omega_{\delta}[\frac{\omega}{\varepsilon}])_{r\bar{r}} = \sum_{\substack{x \bmod \bar{\delta} \\ y \bmod \delta}} \delta_{r,y\bar{\delta}+(x+\omega)\delta}^{(\varepsilon)} \delta_{r,y,\bar{\delta}-(x+\omega)\delta}^{(\varepsilon)} \quad (3.202)$$

where δ is assumed odd (if δ is even then ω should be added to y and not to x) and

$$\delta_{rs}^{(\varepsilon)} = \begin{cases} 0 & \text{if } N \nmid r - s, \\ 1 & \text{if } 2N \mid r - s, \\ (-1)^{2\varepsilon} & \text{if } N \mid r - s \text{ and } 2N \nmid r - s, \end{cases} \quad (3.203)$$

are ε -periodic in N δ -functions (periodic for $\varepsilon = 0$ and antiperiodic for $\varepsilon = 1/2$).

The general modular invariant will have a block diagonal structure labelled by the sectors $[\frac{\varepsilon}{\omega}]$ and the blocks will be

$$\mathcal{M}[\frac{\varepsilon}{\omega}] = \sum c_{\delta} \Omega_{\delta}[\frac{\varepsilon}{\omega}] \quad (3.204)$$

where the sum is over $\delta \mid N$ with δ or $\bar{\delta}$ odd.

The characters of the twisted $N = 2$ can be written as (m is even)

$$\chi_r^T(\tau) = \frac{1}{\Theta_4(\tau)} \Theta_{m/4-r/2, m/4}[\frac{\varepsilon}{1/2}](\tau, z=0), \quad r = 1, \dots, m/2, \quad (3.205)$$

where

$$\varepsilon = \begin{cases} 0 & \text{if } m/4 - r/2 \text{ is even,} \\ \frac{1}{2} & \text{if } m/4 - r/2 \text{ is odd,} \end{cases}$$

and Θ_4 is one of the even classical Jacobi functions. The classical Jacobi functions can be written in terms of (3.197) as follows:

$$\Theta_2 = \Theta\left[\begin{smallmatrix} 1/2 \\ 0 \end{smallmatrix}\right], \quad \Theta_4 = \Theta\left[\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix}\right], \quad \Theta_3 = \Theta\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right], \quad (3.206)$$

which are the even ones, and the odd one:

$$\Theta_1 = \Theta\left[\begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix}\right], \quad (3.207)$$

where $\Theta\left[\begin{smallmatrix} \varepsilon \\ \omega \end{smallmatrix}\right] = \Theta_{\varepsilon,1}\left[\begin{smallmatrix} \varepsilon \\ \omega \end{smallmatrix}\right]$.

When we set $z = 0$ in (3.197) we get

$$\Theta_{\rho, N/2}\left[\begin{smallmatrix} \varepsilon \\ \omega \end{smallmatrix}\right] = \Theta_{-\rho, N/2}\left[\begin{smallmatrix} \varepsilon \\ \omega \end{smallmatrix}\right]. \quad (3.208)$$

This is in agreement with the fact that for the twisted character and the conformal dimensions of the twisted primary fields there is an invariance under $r \rightarrow m-r$. Therefore we can use the invariants (3.204) also for the twisted characters of the N satisfy

$$\mathcal{M}_{s, \bar{s}} = \mathcal{M}_{-s, \bar{s}} = \mathcal{M}_{s, -\bar{s}}, \quad (3.209)$$

which implies that in (3.204) $c_\delta = c_{\bar{\delta}}$. Setting $z = 0$ in what follows let us denote

$$\begin{aligned} \Theta_s^{(1/2)}\left[\begin{smallmatrix} \varepsilon \\ \omega \end{smallmatrix}\right] &= \Theta_{s, m/4}\left[\begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix}\right] / \Theta\left[\begin{smallmatrix} \varepsilon \\ \omega \end{smallmatrix}\right], \\ \Theta_s^{(0)}\left[\begin{smallmatrix} \varepsilon \\ \omega \end{smallmatrix}\right] &= \Theta_{s, m/4}\left[\begin{smallmatrix} \varepsilon \\ \omega \end{smallmatrix}\right] / \Theta\left[\begin{smallmatrix} \varepsilon \\ \omega \end{smallmatrix}\right], \quad \text{with } \left[\begin{smallmatrix} \varepsilon \\ \omega \end{smallmatrix}\right] \neq \left[\begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix}\right], \end{aligned} \quad (3.210)$$

for $m/2$ odd and

$$\Theta_s\left[\begin{smallmatrix} \varepsilon \\ \omega \end{smallmatrix}\right]\left[\begin{smallmatrix} \varepsilon' \\ \omega' \end{smallmatrix}\right] = \Theta_{s, m/4}\left[\begin{smallmatrix} \varepsilon \\ \omega \end{smallmatrix}\right] / \Theta\left[\begin{smallmatrix} \varepsilon' \\ \omega' \end{smallmatrix}\right], \quad (3.211)$$

for $m/2$ even, where the blocks $\left[\begin{smallmatrix} \varepsilon \\ \omega \end{smallmatrix}\right]\left[\begin{smallmatrix} \varepsilon' \\ \omega' \end{smallmatrix}\right]$ are permuted as in (3.147). The modular invariants for $m/2$ odd are given by

$$\sum_{\eta=0, 1/2} \sum_{\text{sectors}} (\Theta^{(\eta)*} \mathcal{M}^{(\eta)} \Theta^{(\eta)})\left[\begin{smallmatrix} \varepsilon \\ \omega \end{smallmatrix}\right] \quad (3.212)$$

where the sum over sectors is for $\left[\begin{smallmatrix} \varepsilon \\ \omega \end{smallmatrix}\right] \neq \left[\begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix}\right]$ and

$$\mathcal{M}^{(0)}\left[\begin{smallmatrix} \varepsilon \\ \omega \end{smallmatrix}\right] = \sum_{\substack{\delta | m/2 \\ \delta \text{ odd}}} c_\delta^{(0)} (\Omega_\delta + \Omega_{\bar{\delta}})\left[\begin{smallmatrix} \varepsilon \\ \omega \end{smallmatrix}\right], \quad (3.213)$$

$$\mathcal{M}^{(1/2)}\left[\begin{smallmatrix} \varepsilon \\ \omega \end{smallmatrix}\right] = \sum_{\substack{\delta|m/2 \\ \delta \text{ odd}}} c_{\delta}^{(1/2)} (\Omega_{\delta} + \Omega_{\bar{\delta}}) \left[\begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix}\right], \quad (3.214)$$

while for $m/2$ even the invariants are given by

$$\sum_{\text{sectors}} (\Theta^* \mathcal{M} \Theta) \left[\begin{smallmatrix} \varepsilon \\ \omega \end{smallmatrix}\right] \left[\begin{smallmatrix} \varepsilon' \\ \omega' \end{smallmatrix}\right] \quad (3.215)$$

where the sum is over the sectors as depicted in (3.147) and

$$\mathcal{M} \left[\begin{smallmatrix} \varepsilon \\ \omega \end{smallmatrix}\right] \left[\begin{smallmatrix} \varepsilon' \\ \omega' \end{smallmatrix}\right] = \sum_{\substack{\delta|m/2 \\ \delta \text{ odd}}} c_{\delta} (\Omega_{\delta} + \Omega_{\bar{\delta}}) \left[\begin{smallmatrix} \varepsilon \\ \omega \end{smallmatrix}\right], \quad \left[\begin{smallmatrix} \varepsilon \\ \omega \end{smallmatrix}\right] \neq \left[\begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix}\right]. \quad (3.216)$$

In order to make contact between the twisted and untwisted parts via (3.193) we note that $\tilde{\chi}_r$ in (3.196) (which are trace over the untwisted Hilbert space but with the automorphism α inserted in the trace and which build up Z_{PT}) can be written as a combination of Θ -functions from $\left[\begin{smallmatrix} 1/2 \\ 0 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix}\right]$. Indeed we have

$$\Theta_{1/2, m/4} \leq \Theta_{s, m} + (-1)^{2\varepsilon} \Theta_{m-s, m}, \quad s \text{ odd}, \quad (3.217)$$

which together with (3.210), (3.211) and (3.196) gives

$$m = 2(\text{mod } 4) : \Theta_{s/2}^{(\varepsilon)} \left[\begin{smallmatrix} 1/2 \\ 0 \end{smallmatrix}\right] = \frac{1}{2} \exp(\pi i(s-1)/2) (\tilde{\chi}_s + (-1)^{2\varepsilon} \tilde{\chi}_{m-s}), \quad (3.218)$$

$$m = 0(\text{mod } 4) : \Theta_{s/2} \leq \left[\begin{smallmatrix} 1/2 \\ 0 \end{smallmatrix}\right] = \frac{1}{2} \exp(\pi i(s-1)/2) (\tilde{\chi}_s - (-1)^{2\varepsilon} \tilde{\chi}_{m-s}). \quad (3.219)$$

For Z_{PT} we can write in case $m = 2(\text{mod } 2)$

$$\begin{aligned} Z_{PT} = & \sum_{\substack{s, \bar{s}(\text{mod } m) \\ \text{odd}}} \frac{e^{\pi i(s-\bar{s})/2}}{4} ((\mathcal{M}^{(0)} + \mathcal{M}^{(1/2)})_{\bar{r}/2, r/2} \left[\begin{smallmatrix} 1/2 \\ 0 \end{smallmatrix}\right] (\tilde{\chi}_{\bar{s}}^* \tilde{\chi}_s + \tilde{\chi}_{m-\bar{s}}^* \tilde{\chi}_{m-s}) \\ & + (\mathcal{M}^{(0)} - \mathcal{M}^{(1/2)})_{\bar{s}/2, s/2} \left[\begin{smallmatrix} 1/2 \\ 0 \end{smallmatrix}\right] (\tilde{\chi}_{\bar{s}}^* \tilde{\chi}_{m-s} + \tilde{\chi}_{m-\bar{s}}^* \tilde{\chi}_s)) \end{aligned} \quad (3.220)$$

and when $m = 0(\text{mod } 4)$

$$\begin{aligned} Z_{PT} = & \sum_{\substack{s, \bar{s}(\text{mod } m) \\ \text{odd}}} \frac{e^{\pi i(s-\bar{s})/2}}{4} ((\mathcal{M} \left[\begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix}\right] + \mathcal{M} \left[\begin{smallmatrix} 1/2 \\ 0 \end{smallmatrix}\right])_{\bar{s}/2, s/2} (\tilde{\chi}_{\bar{s}}^* \tilde{\chi}_s + \tilde{\chi}_{m-\bar{s}}^* \tilde{\chi}_{m-s}) \\ & + (\mathcal{M} \left[\begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix}\right] - \mathcal{M} \left[\begin{smallmatrix} 1/2 \\ 0 \end{smallmatrix}\right])_{\bar{s}/2, s/2} (\tilde{\chi}_{\bar{s}}^* \tilde{\chi}_{m-s} + \tilde{\chi}_{m-\bar{s}}^* \tilde{\chi}_s)). \end{aligned} \quad (3.221)$$

The requirements we have to impose are that Z_{PT} coincides

$$Z_{PT} = \sum_{\substack{s, \bar{s}(\text{mod } m) \\ \text{odd}}} N_{\bar{s}s}(\mathcal{G}) \tilde{\chi}_{\bar{s}}^* \tilde{\chi}_s \quad (3.222)$$

for some $N(\mathcal{G})$ from (3.183) and that Z_{PT} is a quadratic form of the twisted characters X^T with non-negative integer coefficients.

As a result we obtain the A and D_{even} sequences and E_6, E_7, E_8 . When $m = 0 \pmod{4}$ one checks that for Z_{TP} besides the diagonal invariant there is only one nontrivial for $m = 12$, while all the rest of them have negative coefficients. This is consistent with the fact that for $m = 0 \pmod{4}$ both $Z(A_{m-1})$ and $Z(D_{m/2+1})$ produce the same Z_{PT} when χ is changed to $\tilde{\chi}$, namely, $\sum_{r \text{ odd}=1}^{m-1} |\chi_r|^2$, hence by the modular transformations the twisted part of the A sequence of orbifold invariants coincides with the twisted part of the D_{odd} sequence orbifold invariants.

The A invariants are given by $M = \mathbb{1}$, for any m . When $m = 2 \pmod{4}$ there are the even D sequences:

$$M^{(0)} = \mathbb{1}, \quad M^{(1/2)} = 0 \quad (3.223)$$

the E_7 invariant for $m = 18$

$$M^{(0)} \left[\begin{smallmatrix} \varepsilon \\ \omega \end{smallmatrix} \right] = (\Omega_1 + \Omega_9 - \Omega_3) \left[\begin{smallmatrix} \varepsilon \\ \omega \end{smallmatrix} \right], \quad M^{(1/2)} = 0 \quad (3.224)$$

and the E_8 invariant for $m = 30$

$$M^{(0)} \left[\begin{smallmatrix} \varepsilon \\ \omega \end{smallmatrix} \right] = (\Omega_1 + \Omega_{15} - (\Omega_5 + \Omega_3)) \left[\begin{smallmatrix} \varepsilon \\ \omega \end{smallmatrix} \right], \quad M^{(1/2)} = 0. \quad (3.225)$$

When $m = 0 \pmod{4}$ the only nontrivial twisted invariant is at $m = 12$

$$M \left[\begin{smallmatrix} \varepsilon \\ \omega \end{smallmatrix} \right] \left[\begin{smallmatrix} \varepsilon' \\ \omega' \end{smallmatrix} \right] = (\Omega_1 + \Omega_6 - (\Omega_3 + \Omega_2)) \left[\begin{smallmatrix} \varepsilon \\ \omega \end{smallmatrix} \right]. \quad (3.226)$$

More explicitly, for the modular invariant partition function of the $\mathbb{Z}_2$ -orbifold line in the space of $c = 3k/(k+2) = 3(1-2/m)$ theories we obtain

$$Z_{\text{orb},k}(\mathcal{G}; R) = \frac{1}{2} (Z_k(\mathcal{G}; R) + Z_{\text{twist},k}(\mathcal{G})) \quad (3.227)$$

where $\mathcal{G}$ is $A_{m-1}, D_{m/2-1}, E_6, E_7$ or E_8 , $Z_k(\mathcal{G}; R)$ is given in (3.187) and for Z_{twist} we have $(k+2 = m)$

$$\begin{aligned} Z_{\text{twist}}(A_{m-1}) &= \sum_{r \text{ odd}=1}^{m-1} |\tilde{\chi}_r|^2 + 2 \sum_{r=1}^{m/2-1} |\chi_r^T|^2 + |\chi_{m/2}^T|^2 \\ &\quad + 2 \sum_{r=1}^{m/2-1} |\tilde{\chi}_r^T|^2 + |\tilde{\chi}_{m/2}^T|^2 \end{aligned} \quad (3.228)$$

(of course, the terms $\chi_{m/2}^T, \tilde{\chi}_{m/2}^T$ are absent when m is odd);

$$Z_{\text{twist}}(D_{m/2+1(=\text{odd})}) = Z_{\text{twist}}(A_{m-1}), \quad (3.229)$$

$$Z_{\text{twist}}(D_{m/2+1(=\text{even})}) = \sum_{r \text{ odd}=1}^{m/2-2} |\tilde{\chi}_r + \tilde{\chi}_{m-r}|^2 + 2|\tilde{\chi}_{m/2}|^2$$

$$+ 4 \sum_{r \text{ odd}=1}^{m/2-2} |\chi_r^T|^2 + 2|\chi_{m/2}^T|^2 + 4 \sum_{r \text{ odd}=1}^{m/2-2} |\tilde{\chi}_r^T|^2 + 2|\tilde{\chi}_{m/2}^T|^2, \quad (3.230)$$

$$Z_{\text{twist}}(E_6) = |\tilde{\chi}_1 + \tilde{\chi}_7|^2 + |\tilde{\chi}_5 + \tilde{\chi}_{11}|^2 \\ + 2|\chi_4^T|^2 + 2|\chi_1^T + \chi_5^T|^2 + 2|\tilde{\chi}_4^T|^2 + 2|\tilde{\chi}_1^T - \tilde{\chi}_5^T|^2, \quad (3.231)$$

$$Z_{\text{twist}}(E_7) = |\tilde{\chi}_1 + \tilde{\chi}_{17}|^2 + |\tilde{\chi}_5 + \tilde{\chi}_{13}|^2 + |\tilde{\chi}_7 + \tilde{\chi}_{11}|^2 + |\chi_9|^2 \\ + 4|\chi_1^T|^2 + 4|\chi_5^T|^2 + 4|\chi_7^T|^2 + |\chi_9^T|^2 \\ + \tilde{\chi}_9^*(\tilde{\chi}_3 + \tilde{\chi}_{15}) + c.c. + 2\tilde{\chi}_9^*\chi_3^T + c.c. - 2\tilde{\chi}_9^{T*}\tilde{\chi}_3^T + c.c. \\ + 4|\tilde{\chi}_1^T|^2 + 4|\tilde{\chi}_5^T|^2 + 4|\tilde{\chi}_7^T|^2 + |\tilde{\chi}_9^T|^2,$$

$$Z_{\text{twist}}(E_8) = |\tilde{\chi}_1 + \tilde{\chi}_{11} + \tilde{\chi}_{19} + \tilde{\chi}_{29}|^2 + |\tilde{\chi}_7 + \tilde{\chi}_{13} + \tilde{\chi}_{17} + \tilde{\chi}_{23}|^2 \\ + 4|\chi_1^T + \chi_{11}^T|^2 + 4|\chi_7^T + \chi_{13}^T|^2 + 4|\tilde{\chi}_1^T - \chi_{11}^T|^2 + 4|\tilde{\chi}_7 - \tilde{\chi}_{13}|^2. \quad (3.232)$$

Note that the negative signs in the Z_{rr} part of $Z_{\text{twist}}(E)$ cause no problem, because the quadratic form in $\tilde{\chi}$ and $\tilde{\chi}^T$ need not be with positive coefficient. Recall that these are not real characters; they are obtained by insertion of the α automorphism in the trace. The coefficients of the quadratic form going with them do not have the meaning of operator content.

4 Affine Lie (super-)algebras

Summary

The theory and applications of Kac–Moody algebras thrived in the last several decades on the interface between mathematics and physics. Kac–Moody algebras were identified first in the stationary Einstein–Maxwell equations [384], in two-dimensional (2D) principal- (or super-) chiral models [192], in 2D sigma models on coset spaces [113], in 2D Heisenberg model [208], in the (anti-) self-dual sector on pure Yang–Mills theory [114], in supersymmetric ($N = 4$) Yang–Mills theory [565], in completely integrable systems [528], in Toda systems [481] and also as current algebras in two- [570, 124, 439] and three-dimensional [448] models. Naturally, in these pioneer attempts little was used from the representation theory of these algebras (cf. [354] and the references therein). One motivation for our research came from the remarkable papers by the Sato followers [528] which use for the hierarchies of completely integrable systems the most basic representations—the so-called fundamental modules. We recall that every fundamental module appears as the irreducible subquotient of an indecomposable Verma module, the latter being a member of an infinite set of other partially equivalent indecomposable Verma modules. Such a set is called as in the semisimple Lie algebra (SSLA) case a multiplet [153]. However, there are other multiplets besides those containing the fundamental modules. Using an approach developed earlier for semisimple Lie groups or algebras [153] and adapted here for Kac–Moody algebras we classify and parametrize all such multiplets for $A_\ell^{(1)}$ and give them explicitly for $A_1^{(1)}$ and $A_2^{(1)}$. One motivation for this was the search for new hierarchies of completely integrable systems arising from multiplets which do not contain the fundamental modules. This seems very natural since the setting we describe contains infinitely many differential invariant operators; namely, the intertwining operators which give the partial equivalences in the multiplets become differential operators when we realize the HWM as spaces of functions. Every invariant operator between function spaces gives rise to an invariant equation. These invariant operators and equations were very useful as we know from previous chapters and volumes. Such differential equations also led to the successful treatment of some statistical physics models by Belavin, Polyakov and Zamolodchikov [57] by considering another infinite-dimensional Lie algebra, the Virasoro one. We mention this not only because such a line of research is another of our motivations but also because the representation theory of the Virasoro algebra and the Kac–Moody $A_1^{(1)}$ are closely related. The exact correspondence is discussed. A more general relation is that every HWM of a Kac–Moody algebra may be extended to the semidirect product with the Virasoro algebra [533]. The elements of the latter can be realized as bilinear combinations (normally ordered in some sense) of elements of the Kac–Moody algebras, i. e. as elements of the latter universal enveloping algebras. This in turn provides explicit constructions of the Kac–Moody fundamental modules [356]. Thus there is rich ground for interplay between the differential operators invariant under the two algebras. Possible applications of such interplays, e. g. for finding exact formulae for anomalous dimensions and differential equations for the correlation functions were first mentioned in lectures of Polyakov in Shumen (Bulgaria) in August 1984 [496] (see also Todorov [554]). Furthermore, in Knizhnik–Zamolodchikov [392] the simplest one was realized involving a first order linear differential equation. The Kac–Moody algebras encountered by Witten [570] (for even central charge) and by Craigie–Nahm [124] were also good starting points for such applications.

4.1 Multiplet classification of Verma modules over affine Lie algebras and invariant differential operators: the $A_\ell^{(1)}$ example

4.1.1 Definitions and notation

The material in this section is fairly standard and because of earlier availability mostly follows [243] although it may be found in [354] (see also Volume 1).

Let $\mathcal{G}_0$ be a complex finite-dimensional semisimple Lie algebra (SSLA). Let $\mathcal{H}_0$ be the Cartan subalgebra of $\mathcal{G}_0$, and $\langle X, Y \rangle = \text{tr } XY$, $X, Y \in \mathcal{H}_0$ be the Cartan–Killing form. We have the complex infinite-dimensional Lie algebra

$$\hat{\mathcal{G}} = \mathcal{G}_0 \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\hat{c} \quad (4.1)$$

where $\mathbb{C}[t, t^{-1}]$ is the algebra of Laurent polynomials in t . The bracket of $\hat{\mathcal{G}}$ is defined by

$$[X \otimes t^m, Y \otimes t^n] \equiv [X, Y] \otimes t^{m+n} + m\hat{c}\delta_{m,-n}\langle X, Y \rangle \quad (4.2)$$

where $X, Y \in \mathcal{G}_0$, $m, n \in \mathbb{Z}$, $\hat{c}$ is in the centre of $\hat{\mathcal{G}}$. In order to have grading and subsequently a root space decomposition a degree derivation $\hat{d}$ is adjoined to $\hat{\mathcal{G}}$, thus obtaining the *affine Lie algebra* $\mathcal{G}$:

$$\mathcal{G} = \hat{\mathcal{G}} \oplus \hat{d}, \quad \hat{d} \equiv t \frac{d}{dt} \quad (4.3)$$

The Cartan subalgebra $\mathcal{H}$ of $\mathcal{G}$ is therefore

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathbb{C}\hat{c} \oplus \mathbb{C}\hat{d}. \quad (4.4)$$

Furthermore, $\langle, \rangle$ is extended to $\mathcal{G}$ by setting

$$\langle \hat{c}, \hat{d} \rangle = 1, \quad \langle \hat{c}, \hat{c} \rangle = \langle \hat{d}, \hat{d} \rangle = \langle \hat{c}, X \rangle = \langle \hat{d}, X \rangle = 0, \quad X \in \mathcal{H}_0. \quad (4.5)$$

Let us denote by $\Delta_0 = \Delta_0^+ \cup \Delta_0^-$ the root system of $\mathcal{G}_0$, $\Delta_0^\pm$ being the positive and negative roots, respectively. Then the root system $\Delta = \Delta^+ \cup \Delta^-$ of $\mathcal{G}$ ($\pm$ signs are correlated) is

$$\begin{aligned} \Delta^\pm = \{ \beta_{n\kappa\alpha} \equiv n\bar{d} + \kappa\alpha \mid n \in \pm\mathbb{N}, \kappa = 0, 1, -1, \alpha \in \Delta_0^\pm; \\ n = 0, \kappa = \pm 1, \alpha \in \Delta_0^+ \} \end{aligned} \quad (4.6)$$

where $\bar{d}$ is defined by $\bar{d}(\hat{d}) = 1$, $\bar{d}(\hat{c}) = 0 = \bar{d}(X)$, $X \in \mathcal{H}_0$. The corresponding root spaces shall be denoted by $\mathcal{G}_{n\kappa\alpha}^\pm$. The roots with $n = 0$ form $\Delta_0^\pm$, and $\mathcal{G}_{0,1,\alpha}^\pm$ is the root space $\mathcal{G}_\alpha^\pm$ in $\mathcal{G}_0$. The sets of simple roots of Δ_0 and Δ are given by

$$\Delta_{0S} = \{\alpha_1, \dots, \alpha_\ell\}, \quad \Delta_S = \{\alpha_0 \equiv \bar{d} - \tilde{\alpha}, \alpha_1, \dots, \alpha_\ell\} \quad (\ell = \text{rank } \mathcal{G}_0), \quad (4.7)$$

respectively, where $\tilde{\alpha}$ is the highest root of $\mathcal{G}_0$. We introduce the notation for the weights:

$$\Lambda_0 \equiv \lambda_1 \alpha_1 + \cdots + \lambda_\ell \alpha_\ell \in \mathcal{H}_0^*, \quad \Lambda_i \in \mathbb{C}; \quad \Lambda = c\bar{c} + d\bar{d} + \Lambda_0 \in \mathcal{H}^*, \quad c, d \in \mathbb{C} \quad (4.8)$$

where $\bar{c}$ is defined by $\bar{c}(\hat{c}) = 1$, $\bar{c}(\hat{d}) = 0 = \bar{c}(X)$, $X \in \mathcal{H}_0$. Furthermore, $\langle, \rangle$ is extended to $\mathcal{H}^*$ by

$$\langle \bar{c}, \bar{d} \rangle = 1, \quad \langle \bar{c}, \bar{c} \rangle = \langle \bar{d}, \bar{d} \rangle = \langle \bar{c}, \mu \rangle = \langle \bar{d}, \mu \rangle = 0, \quad \mu \in \mathcal{H}_0^*. \quad (4.9)$$

From (4.9) it follows that

$$\langle \beta_{n\kappa\alpha}, \beta_{n'\kappa'\alpha'} \rangle = \kappa\kappa' \langle \alpha, \alpha' \rangle. \quad (4.10)$$

Thus the product of a root for which $\kappa = 0$ with every other root is zero. Such roots are called by Kac *imaginary roots*, while the rest are called *real roots*. Let E_i, F_i, H_i , $i = 1, \dots, \ell$ be the Chevalley generators of $\mathcal{G}_0$:

$$E_i \in \mathcal{G}_{\alpha_i}^+, \quad F_i \in \mathcal{G}_{\alpha_i}^-, \quad H_i = [E_i, F_i], \quad \alpha_i(H_i) = 2. \quad (4.11)$$

We choose $E_0 \in \mathcal{G}_{\tilde{\alpha}}^-, F_0 \in \mathcal{G}_{\tilde{\alpha}}^+$ so that $\tilde{\alpha}(H_0) = -2$, $H_0 = [E_0, F_0]$ and set

$$\begin{aligned} e_0 &\equiv E_0 \otimes t, & f_0 &\equiv F_0 \otimes t^{-1}, & h_0 &\equiv H_0 \otimes 1 + \hat{c}, \\ (e_i, f_i, h_i) &\equiv (E_i, F_i, H_i) \otimes 1, & i &= 1, \dots, \ell. \end{aligned} \quad (4.12)$$

Now we can define an important element $\rho \in \mathcal{H}^*$ [354] (in the SSLA case half the sum of the positive roots):

$$\rho = \rho_c \bar{c} + \rho_d \bar{d} + \rho_1 \alpha_1 + \cdots + \rho_\ell \alpha_\ell, \quad (4.13a)$$

$$\rho(h_a) = \frac{1}{2} \langle \alpha_a, \alpha_a \rangle, \quad a = 0, 1, \dots, \ell. \quad (4.13b)$$

Note that (4.13b) does not fix ρ_d , but this value is not essential and may be set to zero.

4.1.2 Multiplets of reducible Verma modules, imaginary reflections and extended Weyl group

A Verma module V^Λ is a highest weight module completely characterized [354] by its highest weight $\Lambda \in \mathcal{H}^*$ and highest weight vector v such that

$$e_a v = 0, \quad a = 0, 1, \dots, \ell, \quad Xv = \Lambda(X)v, \quad X \in \mathcal{H}. \quad (4.14)$$

As in the SSLA case Verma modules are generically irreducible and again the reducible ones are of special interest. We recall the Kac–Kazhdan criterion [355] according to which V^Λ is reducible only if the condition

$$2\langle \Lambda + \rho, \beta \rangle = m\langle \beta, \beta \rangle \quad (4.15)$$

holds for some $m \in \mathbb{N}$ and a positive root β . In our notation (4.15) may be written for $\beta = \beta_{n\kappa\alpha}$ and $\Lambda = c\bar{c} + d\bar{d} + \Lambda_0$ as

$$2(n(c + \rho_c) + \kappa\langle\Lambda_0 + \rho, \alpha\rangle) = m\kappa^2\langle\alpha, \alpha\rangle. \quad (4.16)$$

Using the same result of [355] we see that whenever (4.15) holds, V^Λ is partially equivalent to $V^{\Lambda'}$, where $\Lambda' = \Lambda - m\beta$. The Verma module $V^{\Lambda'}$ is a submodule of V^Λ . If β is an imaginary root then m is not restricted by (4.15) and $V^{\Lambda'}$ is a submodule of V^Λ for any $m \in \mathbb{N}$, while for $m \in -\mathbb{N}$ then V^Λ is a submodule of $V^{\Lambda'}$. Note that if (4.15) holds for one imaginary root then it holds for all imaginary roots. Note also that if $\beta_{n\kappa\alpha} \in \Delta_0^+$, i. e. $n = 0, \kappa = 1$ then (4.15) reduces to the BGG criterion [70] for SSLA algebras.

It is standard that to every positive real root β there corresponds a reflection which acts in $\mathcal{H}$:

$$w_\beta(\alpha) \equiv \alpha - 2\frac{\langle\alpha, \beta\rangle}{\langle\beta, \beta\rangle}\beta, \quad \langle\beta, \beta\rangle \neq 0. \quad (4.17)$$

These reflections act also in $\mathcal{H}^*$ whenever (4.15) holds as it may be rewritten similarly to (4.17):

$$w_\beta(\Lambda + \rho) \equiv \Lambda + \rho - 2\frac{\langle\Lambda + \rho, \beta\rangle}{\langle\beta, \beta\rangle}\beta = \Lambda + \rho - m\beta = \Lambda' + \rho. \quad (4.18)$$

It is well known [348, 455] that the Weyl group W generated by the simple reflections (corresponding to the simple roots $\alpha_0, \alpha_1, \dots, \alpha_\ell$) is infinite and is equivalent to the semi-direct product $W_0 \times Q^\vee$ of the Weyl group W_0 of $\mathcal{G}_0$ and the dual integer root lattice $Q^\vee$ (the latter being generated by the dual roots $\alpha_i^\vee \equiv \frac{2}{\langle\alpha_i, \alpha_i\rangle}\alpha_i$ of $\mathcal{G}_0$).

What is new in our treatment is that we propose to introduce *imaginary reflections*. These shall be defined on the weights for which $c + \rho_c = 0$ by “l’Hopital’s rule” on (4.18):

$$w_\beta(\Lambda + \rho) \equiv \Lambda + \rho - \beta = \Lambda' + \rho, \quad \langle\beta, \beta\rangle = 0, \quad \langle\Lambda + \rho, \beta\rangle = 0, \quad “m” = 1. \quad (4.19)$$

Now we recall that in the SSLA case the elements of the Weyl group W_0 of $\mathcal{G}_0$ are in 1-to-1 correspondence with the reducible Verma modules contained in multiplets with the following property: each such multiplet contains (exactly) one Verma module for which (4.15) holds for every simple root of $\mathcal{G}_0$, (this weight being a dominant highest weight, see below). Such multiplets may be called maximal since $|\mathcal{M}_0| \leq |W_0|$. It is true that the Kac–Moody Weyl group W corresponds to multiplets with the above property (replace $\mathcal{G}_0$ by $\mathcal{G}$). However, we shall define an extended Weyl group W^e to include also the imaginary reflections introduced above. More precisely, W^e shall correspond to any multiplet which contains one (and then an infinite number of) Verma modules for which (4.15) holds for ℓ simple roots and for the imaginary roots. Examples of W^e are shown below.

Finally, we recall the standard modules. Every reducible Verma module V^Λ contains a unique proper maximal submodule I^Λ so that there is an irreducible HWM

$L(\Lambda) = V^\Lambda / I^\Lambda$ [354]. In the case when Λ is a dominant weight, i. e. $\Lambda(h_i) \in \mathbb{Z}_+$, $i = 0, 1, \dots, \ell$ (which implies (4.15) holding for every real $\beta \in \Delta^+$), $L(\Lambda)$ is called a *standard module* (Kac calls it *integrable module*). An equivalent definition (usually used) of a standard module is as irreducible HWM with dominant weight Λ and such that $(f_i)^{\Lambda(h_i)+1}v = 0$, $i = 0, 1, \dots, \ell$. We invoked both definitions since this makes the place of the standard modules clear in the set of HWM. (It is clear that standard modules correspond to finite-dimensional modules in the SSLA case.) From the standard modules the *fundamental modules* mostly used, such that $\Lambda_i(h_j) = \delta_{ij}$, $i, j = 0, 1, \dots, \ell$. Λ_0 is called the *basic module*.

4.1.3 Multiplet classification in the $A_\ell^{(1)}$ case

From now on we restrict ourselves to the affine Lie algebra $\mathcal{G} = A_\ell^{(1)}$, i. e. when $\mathcal{G}_0 = \mathfrak{sl}(\ell + 1, \mathbb{C})$. In the notation of (4.7) $\tilde{\alpha} = \alpha_1 + \dots + \alpha_\ell$ and the Cartan matrix (a_{ij}) has the well-known form

$$a_{ij} \equiv 2 \frac{\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_i, \alpha_i \rangle} = \langle \alpha_i, \alpha_j \rangle = a_{ji} = 2\delta_{ij} - \delta_{i+1,j} - \delta_{i0}\delta_{j\ell}, \quad 0 \leq i \leq j \leq \ell. \quad (4.20)$$

Consequently the element ρ (4.13a) is fixed using (4.13b) as

$$\rho_c = \ell + 1, \quad \rho_i = i(\ell + 1 - i)/2, \quad i = 1, \dots, \ell. \quad (4.21)$$

Furthermore, we introduce the signature $\chi = \chi(\Lambda)$ of the highest weight $\Lambda = c\bar{c} + d\bar{d} + \Lambda_0$

$$\chi = [d, m_0, m_1, \dots, m_\ell], \quad m_i = (\Lambda + \rho)(h_i), \quad i = 0, 1, \dots, \ell \quad (4.22)$$

Note that if $m_i \in \mathbb{N}$ for $i = 1, \dots, \ell$ then the set of nonnegative integers $(r_1, \dots, r_\ell)$, $r_i \equiv m_i - 1$, determines a finite-dimensional representation of $\mathfrak{sl}(\ell + 1, \mathbb{C})$ and also a UIR of $\mathfrak{su}(\ell + 1)$. Instead of m_0 we shall also use $c = m_0 + m_1 + \dots + m_\ell - \rho_c$; c is a convenient quantity having the same value for two partially equivalent HWM since c belongs to the centre of $\mathcal{G}$. The reducibility conditions (4.15), recalling that for every positive root $\alpha = \alpha_{ij} = \alpha_i + \dots + \alpha_j$, $1 \leq i \leq j \leq \ell$, now becomes

$$n(c + \rho_c) + \kappa(m_i + \dots + m_j) = m\kappa^2. \quad (4.23)$$

The signature of the partially equivalent representation $\Lambda' = \Lambda - m\beta_{n\kappa\alpha_{ij}}$ is given by

$$\chi' = \chi(\Lambda') = [d', m'_0, m'_1, \dots, m'_\ell], \quad (4.24a)$$

$$d' = d - mn, \quad m'_0 = m_0 + m\kappa(\delta_{i0} + \delta_{j\ell}), \quad (4.24b)$$

$$m'_k = m_k - m\kappa(\delta_{ki} + \delta_{kj} - \delta_{k,i-1} - \delta_{k,j+1}), \quad k = 1, \dots, \ell,$$

$$m = n(c + \rho_c) + \kappa(m_i + \dots + m_j) \quad \text{for } \kappa \neq 0; \quad m \in \mathbb{Z} \setminus \{0\} \quad \text{for } \kappa = 0.$$

We shall divide the reducible Verma modules into two (obvious) groups: (i) *nonsingular Verma modules*, for which (4.15) is satisfied only for some real root(s), i. e. $c + \rho_c \neq 0$; (ii) *singular Verma modules* for which (4.15) is satisfied for the imaginary roots, i. e. $c + \rho_c = 0$. The exact results first classify the types of multiplets and then parametrize the various types.

Theorem 1. *The types of multiplets of nonsingular reducible Verma modules of $A_\ell^{(1)}$ are in 1-to-1 correspondence with the following sets of nonnegative integers:*

$$\begin{aligned}\tau_\ell &= \{p; v; r; L_r; P_r; Q_r; L'_1, \dots, L'_{r+1}\} \\ L_r &= \{\ell_1, \dots, \ell_r\}, \quad L_0 = \emptyset, \quad P_r = \{p_1, \dots, p_{r+1}\}, \\ Q_r &= \{q_1, \dots, q_{r+1}\}, \quad L'_i = \{\ell_{i1}, \dots, \ell_{iq_i}\},\end{aligned}\tag{4.25}$$

which satisfy the following conditions:

$$\begin{aligned}1 \leq p \leq \ell + 1; \quad 0 \leq v \leq \delta_{p, \ell+1}; \quad 0 \leq r \leq \min(p-1, \ell-p) + \delta_{p, \ell+1}; \\ 1 \leq \ell_1 \leq \dots \leq \ell_r \leq \ell_{r+1} \equiv p - \ell_1 - \dots - \ell_r; \quad 1 \leq p_i \leq \ell_i, \\ 0 \leq q_i \leq \min(p_i - 1, \ell_i - p_i - \delta_{r0} \delta_{p-1, p-1}), \\ 1 \leq \ell_{i1} \leq \dots \leq \ell_{iq_i} \leq \ell_{i, q_i+1} \equiv p_i - \ell_{i1} - \dots - \ell_{iq_i}, \quad i = 1, \dots, r+1; \\ \ell_i = \ell_{i+1} \Rightarrow p_i \leq k_{i+1}.\end{aligned}\tag{4.26}$$

For every nonsingular reducible Verma module V^Λ there exists exactly one type τ_ℓ such that V^Λ is contained in a multiplet of that type. $\diamond$

Remark 1. Fix j , $1 \leq \ell - 1$. Then the types τ_j of multiplets of nonsingular reducible Verma modules of $A_j^{(1)}$ are in 1-to-1 correspondence with the types τ_ℓ of $A_\ell^{(1)}$ (4.25) for which ℓ is replaced by j in (4.26). $\diamond$

Proposition 1. Fix j , $1 \leq \ell$. Then the types of reducible Verma modules of $\mathfrak{sl}(j+1, \mathbb{C})$, to be denoted by τ_{0j} , are in 1-to-1 correspondence with the types τ_ℓ of $A_\ell^{(1)}$ (4.25) for which in (4.26) the restrictions on p and q_i are replaced by $1 \leq p \leq j$, and $0 \leq q_i \leq \min(p_i - 1, \ell_i - p_i)$. $\diamond$

Before formulating the parametrization theorem we recall [354] that the expression the $w = w_{i_1} \dots w_{i_u} \in W$ is called reduced if it is the minimal possible among all representations of w as a product of simple reflections $w_i \in W$. Note that for many elements of W the reduced expression is not unique.

Theorem 2. *Every multiplet of nonsingular reducible Verma module V^Λ over $A_\ell^{(1)}$ of type τ_ℓ , cf. (4.25), is parametrized by one arbitrary complex number d , $\ell + 1 - p$ non-integer complex numbers and by $p_1 + \dots + p_{r+1}$ positive integers as follows. Consider the signature $\chi = \chi(\Lambda^*) = [d, m_0, m_1, \dots, m_\ell]$ where*

$$\begin{aligned}m_{t(i)}, \quad i = 1, \dots, r+1; \quad t(i) = \ell_1 + \dots + \ell_i + i, \\ m_j, \quad j = p + r + 2, \dots, \ell + 1 \quad (m_{\ell+1} \equiv m_0),\end{aligned}\tag{4.27}$$

are the $\ell + 1 - p$ noninteger complex numbers;

$$\begin{aligned} (-1)^v m_k, \quad k = t_i(s-1) + 1, \dots, t_i(s) - 1, \\ t_i(s) = t(i-1) + \ell_{i1} + \dots + \ell_{is} + s, \\ s = 1, \dots, q_i + 1; \quad i = 1, \dots, r + 1; \quad (t_i(0) \equiv t(i-1); \quad t(0) \equiv 0), \end{aligned} \quad (4.28)$$

are the $p_1 + \dots + p_{r+1}$ positive integers; all other m_i in χ are zero. The other Verma modules in the multiplet are in 1-to-1 correspondence with those elements w of the Weyl group W (if $p = \ell + 1$) or of W_0 (if $1 \leq p \leq \ell$) which do not have a reduced expression $w = w_{i_1} \dots w_{i_u}$ such that m_{i_u} is zero, and which (if $1 \leq p \leq \ell$) do have a reduced expression such that $t(i-1) + 1 \leq i_1, \dots, i_u \leq t(i) - 1$, for a fixed (depending on w) $i = 1, \dots, r + 1$. For such w the corresponding $\Lambda = \Lambda(w)$ is such that $\Lambda = w\Lambda^*$, if $v = 0$, and $\Lambda^* = w\Lambda$, if $v = 1$.

Corollary 1. The statement of Theorem 2 remains true if we replace $A_\ell^{(1)}$ with $\mathfrak{sl}(\ell + 1, \mathbb{C})$ with the following changes: parametrization is by $\ell - p$ noninteger complex numbers and $p_1 + \dots + p_{r+1}$ positive integers; $\chi \rightarrow \chi_0 = [m_1, \dots, m_\ell]$; in (4.27) $j = p + r + 2, \dots, \ell$. $\diamond$

The proof of all statements above may be sketched as follows. There is 1-to-1 correspondence between the types τ_ℓ in Theorem 1 and the HW Λ^* in Theorem 2. The types for which $p_i = \ell_i$, $i = 1, \dots, r + 1$, which we may call principal multiplet types, correspond to Λ^* for which all m_i are nonzero (since $p_1 + \dots + p_{r+1} = p$) and represent all partitions of the number p into positive ordered summands (ordered because of Weyl group permutations). These summands are then used to enumerate the p positive integers in (4.28) and $k = 1, \dots, p_1, p_1 + 2, \dots, p_1 + p_2 + 1, p_1 + p_2 + 3, \dots, p$; the enumeration separates each summand from the next by omission of one number which is necessary to distinguish between different partitions and is sufficient to distinguish different Λ^* . The latter follows from the fact that the Weyl group cannot permute a noninteger entry in the signature (the ones referred to are given by the first line in (4.27)). (One should keep in mind that the entry m_0 is next to m_1 and m_ℓ .) Actually the elements of W which may act on Λ^* generate a subgroup W^* of W which is equivalent to the direct product $W(\ell_1) \times \dots \times W(\ell_{r+1})$, where $W(\ell_i)$ is generated by the simple reflections w_j , $t(i-1) + 1 \leq j \leq t(i) - 1$, and is equivalent to the Weyl group W_0 of $\mathfrak{sl}(\ell_i + 1, \mathbb{C})$ if $p \neq \ell + 1$. The types for which $p_i \neq \ell_i$ for at least one i shall be called *principal types reductions* as they correspond to the different (Weyl-) distinguishable ways of replacing some of the positive integers by zeroes (at least one remains nonzero), which in turn amounts to the description of all partitions of p_i into q_i positive ordered summands. The difference in the above consideration is that here the “acting” Weyl group W^* is not further divided into more factors but the factors are themselves diminished (cf. below for examples). Finally, also in connection with Proposition 1 and Corollary 1, we notice that the value of d is not essential for distinguishing the different types. $\square$

Remark 2. As is clear from Theorem 2 and Remark 1 for $A_j^{(1)}$ only multiplets with $p = j + 1$ are infinite, although according to Proposition 1 and Corollary 1 they correspond to the finite multiplets of $\mathfrak{sl}(2 + j, \mathbb{C})$ with $p = j + 1$. $\diamond$

Remark 3. All statements above follow the multiplet classification approach proposed in [153]. One should note that although the results for $\mathfrak{so}(p, q)$ there look very complicated they do not distinguish between principal types and their reductions. $\diamond$

(ii) Next we turn to the classification of the singular reducible Verma modules.

Theorem 3. The types τ_ℓ^s of multiplets of singular Verma modules over $A_\ell^{(1)}$ are in 1-to-1 correspondence with the types of τ_ℓ over $A_\ell^{(1)}$, formulae (4.25) and (4.26), for which the restrictions for p, p_i and q_i are replaced by $1 \leq p \leq \ell, 1 - \delta_{r0} \leq p_i \leq \ell_i, 0 \leq q_i \min(p_i - 1 + \delta_{r0}, \ell_i - p_i)$. For every singular reducible Verma module V^Λ over $A_\ell^{(1)}$ there exists exactly one type τ_ℓ^s such that V^Λ is contained in a multiplet of that type. $\diamond$

Theorem 4. Every multiplet of singular reducible Verma module over $A_\ell^{(1)}$ of type τ_ℓ^s is parametrized by one arbitrary complex number d , $\ell - p$ noninteger complex numbers and $p_1 + \dots + p_r$ positive integers as follows. Consider the singular Verma module V^Λ with signature $\chi = \chi(\Lambda) = [d, \tilde{m}_0, m_1, \dots, m_\ell]$, $\tilde{m}_0 = -m_1 - \dots - m_\ell$. Then m_i are given by (4.27) (with $j \leq j$) or (4.28) (with $v = 0$). The other Verma modules $V^{\Lambda'}$ are in 1-to-1 correspondence with those elements $w \times d^n$, $n \in \mathbb{Z}$, of the extended Weyl group $W^e = W_0 \times Q^d$ for which w fulfils the conditions in Theorem 2 (where Q^d is the integer lattice generated by the imaginary reflections (4.19) for $\beta = d$). Then $\Lambda' = (w \times d^n)\Lambda = w\Lambda^n$, where $\chi(\Lambda^n) = [d - n, m_0, m_1, \dots, m_\ell]$. $\diamond$

The proofs of Theorems 3 and 4 follow the ideas of the proof sketched above. $\square$

Below the examples $\ell = 1, 2$ are considered in detail.

4.1.4 The $A_1^{(2)}$ case

According to Theorem 1 we have five nonsingular types: $p = \ell_1 = 1, 2; 1 \leq p_1 \leq \ell_{11} \leq p; 0 \leq v \leq \delta_{p2}; r = q_1 = 0$ in all cases. (Below $\Lambda = \Lambda^*$ of Theorem 2)

The type with $p = 1$ is the well-known $\mathfrak{sl}(2, \mathbb{C})$ multiplet (cf. Volume 1) consisting of two members with HW Λ and $\Lambda - m\alpha_1$, $m \in \mathbb{N}$, and $V^{\Lambda - m\alpha_1}$ is a submodule of V^Λ .

The type $p = p_1 = 2, v = 0$, may be described by the following diagram:

$$\begin{array}{ccccccc}
 & & \Lambda'_{00} & \xrightarrow{w_1} & \Lambda_{01} & \xrightarrow{w_1} & \Lambda_{0k} & \xrightarrow{w_0} & \Lambda'_{0k} & \xrightarrow{w_1} & \dots \\
 & \nearrow^{w_0} & & \nearrow^{w_1} & & \nearrow^{w_1} & & \nearrow^{w_0} & & \nearrow^{w_1} & \\
 \Lambda & & & & & & & & & & \\
 & \searrow_{w_1} & & \searrow_{w_0} & & \searrow_{w_0} & & \searrow_{w_1} & & \searrow_{w_0} & \\
 & & \Lambda'_{10} & \xrightarrow{w_0} & \Lambda_{11} & \xrightarrow{w_0} & \Lambda_{1k} & \xrightarrow{w_1} & \Lambda'_{1k} & \xrightarrow{w_0} & \dots
 \end{array} \quad (4.29)$$

where the signatures $\chi_{ak}^{(l)}$, $a = 0, l, k = 0, 1, \dots$, of the HW $\Lambda_{ak}^{(l)}$ are

$$\begin{aligned}
 \chi_{0k} &= [d - k(km_0 + (k-1)m_1), (2k+1)m_0 + 2km_1, \\
 &\quad -2km_0 - (2k-1)m_1],
 \end{aligned} \quad (4.30a)$$

$$\begin{aligned} \chi'_{0k} = [d - (k+1)((k+1)m_0 + km_1), -(2k+1)m_0 - 2km_1, \\ 2(k+1)m_0 + (2k+1)m_1], \end{aligned} \quad (4.30b)$$

$$\begin{aligned} \chi_{1k} = [d - k(km_0 + (k+1)m_1), -(2k-1)m_0 - 2km_1, \\ 2km_0 + (2k+1)m_1], \end{aligned} \quad (4.30c)$$

$$\begin{aligned} \chi'_{1k} = [d - k(km_0 + (k+1)m_1), (2k+1)m_0 + 2(k+1)m_1, \\ -2km_0 - (2k+1)m_1]. \end{aligned} \quad (4.30d)$$

Note $\chi_{00} = \chi_{10} = [d, m_0, m_1]$. Note that the sum of the second and third entry in every χ equals $m_0 + m_1 = c + \rho_c$. We have indicated the intertwining maps corresponding to the simple reflections w_0, w_1 (or equivalently to the simple roots α_0, α_1) and we use the convention that arrows point to the embedded Verma modules. It is obvious that

$$\Lambda_{0k} = (w_1 w_0)^k \Lambda, \quad \Lambda_{1k} = (w_0 w_1)^k \Lambda, \quad \Lambda'_{0k} = w_0 (w_1 w_0)^k \Lambda, \quad \Lambda'_{1k} = w_1 (w_0 w_1)^k \Lambda \quad (4.31)$$

and that $w_0(w_1 w_0)^k, w_1(w_0 w_1)^k$ correspond to the positive real roots

$$\beta_{k+1,-1}, \beta_{k,1} \quad \text{where } \beta_{nk} \equiv \beta_{n\kappa\alpha_1} = n\bar{d} + \kappa\alpha_1. \quad (4.32)$$

The dashed arrows in (4.29) starting from $\Lambda_{0k}, \Lambda'_{0k}, \Lambda_{1k}, \Lambda'_{1k}$ correspond (respectively) to the roots:

$$\beta_{2k,1}, \beta_{2k+2,-1}, \beta_{2k+1,-1}, \beta_{2k+1,1}. \quad (4.33)$$

We should mention that this type is also described in [515]. The authors of this reference display the Weyl group W but do not introduce the notion of multiplets of Verma modules. They call (4.29) “the Bruhat ordering diagram” of W .

We recall that the embedding of a Verma modules $V^{\Lambda'}$ in another reducible Verma module V^Λ is equivalent to the existence in V^Λ of at least one vector v_s different from the highest weight vector v of V^Λ and having the weight of the HWV v' of $V^{\Lambda'}$ [355]. This is in contrast with the uniqueness of v_s in the SSLA case and is the cause of the non-uniqueness above. The vector v_s —usually called *singular vector*—may be expressed as a homogeneous polynomial $\mathcal{P}(f_0, f_1, \dots, f_\ell)$ in the negative canonical generators (cf. (4.12)) acting on v . As in the SSLA case [150] the degrees of homogeneity depend on the root under which V^Λ is reducible, namely [154] the degrees in $f_0, f_1, \dots, f_\ell$, respectively, are

$$\begin{aligned} mn, m(n + \kappa k_1), \dots, m(n + \kappa k_\ell); \\ m = 2 \frac{\langle \Lambda + \rho, \beta \rangle}{\langle \beta, \beta \rangle}, \quad \beta = \beta_{n\kappa\alpha}, \quad \alpha^\vee = k_1 \alpha_1 + \dots + k_\ell \alpha_\ell. \end{aligned} \quad (4.34)$$

For example, if $\beta = \alpha_i$ is a simple root we have

$$v_s = \text{const} \cdot f_i^m v_0. \quad (4.35)$$

Going back to the $A_1^{(1)}$ case we first note that the homogeneity degrees in f_0, f_1 of the maps represented by the solid arrows in (4.29) and starting from $\Lambda_{0k}, \Lambda'_{0k}, \Lambda_{1k}, \Lambda'_{1k}$, respectively, are

$$\begin{aligned} &((2k+1)m_0 + 2km_1, 0), (0, 2(k+1)m_0 + (2k+1)m_1), \\ &(0, 2km_0 + (2k+1)m_1), ((2k+1)m_0 + 2(k+1)m_1, 0), \end{aligned} \quad (4.36)$$

i. e. they are determined by the positive entries in the signatures. The homogeneity degrees of the maps corresponding to the dashed arrows in (4.29) in the four cases as above are

$$\begin{aligned} &(2km_1, (2k+1)m_1), (2(k+1)m_1, (2k+1)m_1), \\ &((2k+1)m_0, 2km_0), ((2k+1)m_0, 2(k+1)m_0). \end{aligned} \quad (4.37)$$

Remark 4. The general structure of these singular vectors was given in [436]. Before that there were known some simple examples [154]. For example, the singular vector corresponding to dashed arrow starting from Λ'_{00} in the case $m_1 = 1$ is

$$v_{\beta_{2,-1}} = [(m_0 + 1)(m_0 + 2)f_0^2 f_1 - 2m_0(m_0 + 2)f_0 f_1 f_0 + m_0(m_0 + 1)f_1 f_0^2]v. \quad (4.38)$$

The operator starting from Λ'_{10} is obtained from (4.38) by the changes $m_0 \leftrightarrow m_1, f_0 \leftrightarrow f_1$. The dashed arrow starting from Λ_{01} (not shown in (4.29)) corresponds to the root $\beta_{2,1}$. In the case $m_0 = m_1 = 1$ the singular vector is given by

$$\begin{aligned} v_{\beta_{2,1}} = & [f_0^2 f_1^3 + 2f_1^3 f_0^2 - 3f_0 f_1^3 f_0 + 6f_1 f_0 f_1^2 f_0 \\ & + 3f_0 f_1^2 f_0 f_1 - 6f_1^2 f_0 f_1 f_0 - 3f_0 f_1 f_0 f_1^2]v. \end{aligned} \quad (4.39) \quad \diamond$$

The type $p = p_1 = 2, v = 1$ is obtained from the above by changing m_0, m_1 into negative and turning the arrows in the opposite direction.

The types $p = 2, p_1 = 1, v = 0, 1$ are obtained from those with $p_1 = 2$ above by setting $m_1 = 0$ (or equivalently $m_0 = 0$). Then one notices that $\Lambda_{0k} = \Lambda'_{1k}$ and $\Lambda_{1,k+1} = \Lambda'_{0k}$. Thus, (4.29) becomes a simple chain. Explicitly, we have

$$\chi_{0k} = [d - k^2 m_0, (2k+1)m_0, -2km_0], \quad (4.40a)$$

$$\chi_{1k} = [d - k^2 m_0, -(2k-1)m_0, 2km_0], \quad (4.40b)$$

$$\Lambda = \Lambda_{00} = \Lambda_{10} \xrightarrow{w_0} \Lambda_{11} \xrightarrow{w_0} \Lambda_{01} \xrightarrow{w_0} \cdots \xrightarrow{w_0} \Lambda_{1k} \xrightarrow{w_0} \Lambda_{0k} \xrightarrow{w_0} \cdots \quad (4.41)$$

From Theorem 3 we obtain one singular type of multiplets characterized by $d \in \mathbb{C}$ and $m \in \mathbb{N}$ and another one (the reduction of the first) fixed by $d \in \mathbb{C}$. The first type of multiplets includes all Verma modules which are reducible w. r. t. the imaginary roots and w. r. t. the positive roots with one and the same m in (4.15). Explicitly, these are the Verma modules $\mathcal{M}(\Lambda_0^+ - n\bar{d} - \varepsilon m\alpha_1), \Lambda_0^+ = (m-1)\Lambda_1 - (m+1)\Lambda_2, m \in \mathbb{N}, n \in \mathbb{Z}, \varepsilon = 0, 1$. The

multiplets are presented by such a diagram as appeared first in [154] (equations (35) and (36)):

$$\begin{array}{ccccccc}
 \cdots & \xrightarrow{\bar{d}} & \Lambda + \bar{d} & \xrightarrow{\bar{d}} & \Lambda & \xrightarrow{\bar{d}} & \Lambda - \bar{d} & \xrightarrow{\bar{d}} & \cdots \\
 & & \downarrow w_1 & & \downarrow w_1 & & \downarrow w_1 & & \\
 \cdots & \xrightarrow{\bar{d}} & \Lambda + \bar{d} - m\alpha_1 & \xrightarrow{\bar{d}} & \Lambda - m\alpha_1 & \xrightarrow{\bar{d}} & \Lambda - \bar{d} - m\alpha_1 & \xrightarrow{\bar{d}} & \cdots
 \end{array} \tag{4.42}$$

where the signature of Λ_0^+ is $\chi = [d, -m, m]$.

Example. We give also the expression for the singular vectors corresponding to the imaginary reflections:

$$v_d = [(m' + 1)f_0f_1 + (1 - m')f_1f_0]v \tag{4.43}$$

where $m' = m$ for those on the first row of (4.42), while $m' = -m$ for those on the second row of (4.42). Note that when $m' = 1$ then v_d is the composition of f_1 and f_0 . $\diamond$

The other (reduction) singular type is obtained from the first by setting $m = 0$ and is given by the first row of (4.42); (4.43) is valid with $m' = 0$.

The singular types for $A_1^{(1)}$ are considered in detail in Section 4.2.

4.1.5 The Virasoro- $A_1^{(1)}$ correspondence

The correspondence

Even now it is not clear what the precise connection is between the full representation theory of the Virasoro algebra, of Kac-Moody algebras (although an analogy was drawn in [515]) and of the semidirect product. Here we make precise and explicit the analogy between the multiplet classification of the reducible Verma modules of the Kac-Moody algebra $A_1^{(1)}$ and the analogous classification for Virasoro algebra given in Section 3.3.

Below we use results of Chapter 3 where we considered the Virasoro algebra and $N = 1$ super-Virasoro algebras in a unified fashion using the parameters μ, ν to distinguish these three cases. Here we restrict ourselves to the Virasoro algebra, which means that here we use the values $\mu = 0, \nu = 1$ for the formulae we take from Chapter 3.

We start the comparison between the most prominent types, namely, type N_-^1 for Virasoro and the $A_1^{(1)}$ type (2,2,0) (i. e. with $p = p_1 = 2, \nu = 0$). From the diagrams of these types given in Table 3.1 (Section 3.3) and (4.29), respectively, it is clear that there should be some correspondence.

However, for exactness we should also address a correspondence of the singular vectors. We recall that for affine Lie algebras (as for SSLA) any singular vector v_s may be expressed as a homogeneous polynomial $\mathcal{P}(f_0, f_1, \dots, f_\ell)$ in the negative canonical

generators acting on the highest weight vector v . Here $\ell = \text{rank } \mathcal{G}_0 = 1$ and the degrees of homogeneity in f_0, f_1 are clear from (4.30).

Furthermore, it turns out [155] that to establish the exact correspondence we should consider the “enlarged” diagrams in which the *primed* entries should be omitted. Namely, in the $A_1^{(1)}$ case instead of diagram (4.29) we should consider the following diagram:

$$\begin{array}{ccccccc}
 & & \Lambda_{01} & \xrightarrow{\hat{w}_0} & \Lambda_{02} & \cdots & \xrightarrow{\Lambda_{0k}} & \xrightarrow{\frac{\hat{m}(2k+1)-m_1}{2\hat{m}(k+1)-m_1}} & \Lambda_{0,k+1} & \cdots \\
 \Lambda & \swarrow & & \searrow & & & \swarrow & \searrow & & \\
 & \hat{w}_1 & & \hat{w}_1 & & & \hat{w}_1 & & & \\
 & \searrow & & \swarrow & & & \searrow & \swarrow & & \\
 & & \Lambda_{11} & \xrightarrow{\hat{w}_1} & \Lambda_{12} & \cdots & \xrightarrow{\Lambda_{1k}} & \xrightarrow{\frac{\hat{m}(2k+1)+m_1}{2\hat{m}k+m_1}} & \Lambda_{1,k+1} & \cdots
 \end{array} \quad (4.44)$$

where $\hat{m} \equiv m_0 + m_1$, and we have indicated the Weyl group elements $\hat{w}_0 \equiv w_1 w_0$ and $\hat{w}_1 \equiv w_0 w_1$ appearing due to the enlarged diagram. Furthermore, we have shown the homogeneity in f_0, f_1 of the embedding maps (which are compositions of two singular vectors) between the pairs of Verma modules with HW $\Lambda_{0k}, \Lambda_{0,k+1}$ and $\Lambda_{1k}, \Lambda_{1,k+1}$. (In order to avoid clutter the degree in f_0 is shown above the embedding map, and the degree in f_1 is shown below the embedding map.) Recalling that $\Lambda = \Lambda_{00} = \Lambda_{10}$ we see that the homogeneity between the Verma modules with HW Λ, Λ_{01} is $m_0, 2m_0 + m_1$, while the homogeneity between the Verma modules with HW Λ, Λ_{11} is $2m_1 + m_0, m_1$.

Analogously, in the Virasoro case the “enlarged” diagram (instead of the one in Table 3.1) is

$$\begin{array}{ccccccc}
 & & V_{01} & \xrightarrow{\quad} & V_{02} & \cdots & \xrightarrow{V_{0k}} & \xrightarrow{pq(2k+1)+mq-np} & V_{0,k+1} & \cdots \\
 V & \swarrow & & \searrow & & & \swarrow & \searrow & & \\
 & & V_{11} & \xrightarrow{\quad} & V_{12} & \cdots & \xrightarrow{V_{1k}} & \xrightarrow{pq(2k+1)-mq+np} & V_{1,k+1} & \cdots
 \end{array} \quad (4.45)$$

Recalling that $V = V_{00} = V_{10}$ we see that the degree of the singular vector between the Verma modules V, V_{01} is $pq + mq - np$, while the homogeneity between the Verma modules V, V_{11} is $pq - mq + np$.

Now if we compare (3.30), (4.30), (4.44), and (4.45) we immediately get the correspondence between the parameters in the signature $\chi = [d, m_0, m_1]$ which distinguish the type $(2,2,0)$ multiplets of $A_1^{(1)}$ and the parameters (p, q, m, n) which distinguish the type N_-^1 multiplets of the Virasoro algebra (cf. equation (32) of [155]):

$$[d, m_0, m_1] = \eta[-h_{mn}, pq + mq - np, np - mq] \quad (4.46)$$

where η is a common rational multiple possibly depending on (m, n, p, q) . Thus, we have established the correspondence needed between these types.

Another case in which the correspondence holds involves $A_1^{(1)}$ type $(2, 1, 0)$ (i. e. with $p = 2, p_1 = 1, v = 0$), which is obtained from the type $(2, 2, 0)$ by setting $m_1 = 0$ (or $m_0 = 0$). Then, as explained, there remains a single line of embeddings given by (4.40) and (4.41). On the Virasoro side this corresponds to the type N_-^{22} with $\varepsilon = 0, np = mq$, (3.39a):

$$V_k^0 = V_k^{h_0^0, c}, \quad h_k^0 = h_0^0 + qpk^2, \quad h_0^0 = -\frac{(p-q)^2}{4pq}. \quad (4.47)$$

Thus, the analogue of (4.46) is

$$[d, m_0, 0] = \eta \left[-h_{pq} = \frac{(p-q)^2}{4pq}, pq, 0 \right]. \quad (4.48)$$

We note that this $A_1^{(1)}$ type accommodates also the Virasoro types N_-^{23} when $c = 1$. Explicitly, in the case $\varepsilon = 0$ one sets $p = q = 1$ in (4.47), then (4.48) gives

$$[d, m_0, 0] = \eta[0, 1, 0]. \quad (4.49)$$

Finally, we note that the conjugated $A_1^{(1)}$ types $(2, 2, 1)$ and $(2, 1, 1)$ when all embedding arrows are inverted, correspond to Virasoro types N_+^1 and N_+^{22}, N_+^{23} . The analogue of (4.46) is [155]

$$[d, m_0, m_1] = \eta[-h_{mn}^+, -pq - mq + np, -np + mq], \quad (4.50)$$

while the analogues of (4.48) and (4.49) follow obviously.

Applications of the correspondence

We recall that if the Virasoro HWMs are realized as function spaces the embedding maps between them are represented by invariant differential operators. The construction of such a differential operator obviously consists of two steps: Find the corresponding singular vector expressed for more convenience in terms of L_{-1} and L_{-2} . Next, find the differential operators corresponding to L_{-1} and L_{-2} and substitute them in the expression for the singular vector obtained in the first step. Naturally the second step depends heavily on the function space realization implemented.

In the applications [57] are used the Virasoro type N_-^1 multiplets. There is 1-to-1 correspondence between these multiplets and the primary fields entering a “minimal theory” [57] which is fixed by the value of c and thus of p and q . The multiplet with $n = q - 1, m = p - 1$, corresponds to the identity operator, since $h_{q-1, p-1} = h_{1,1} = 0$.

Furthermore, for instance the critical point of the two-dimensional Ising model is described in [57] by a minimal conformal theory with $c = \frac{1}{2}$ (thus $p/q = 3/4$) containing three fields: the identity operator I (corresponding to the irreducible HWM on the tip of the multiplet with $(m, n) = (2, 3)$); the local spin σ (with $(m, n) = (1, 2)$) and the energy

density ϵ ($(m, n) = (1, 3)$). From (4.46) with $\eta = 1$ we obtain the corresponding values of $m_1 = 1, 2, 5$. As the latter is related to the $SU(2)$ spin j : $m_1 = 2j + 1$, we thus correctly recover the spins of the three fields: $j = 0, \frac{1}{2}, 2$.

More generally, let us have a field $\Phi_{m,n}$ in such theory with Virasoro parameters p, q so that $qm \langle pn, n \rangle q$. Then we have for the $SU(2)$ dimension and spin:

$$\dim \Phi_{m,n} = pn - mq = 2j_{mn} + 1. \quad (4.51)$$

Of course, this theory also carries a Kac–Moody $A_1^{(1)}$ representation, the central element taking the value

$$c' = pq - 2 = m_0 + m_1 - 2, \quad (4.52)$$

which follows from (4.46) with $\eta = 1$ and recalling that $m_0 + m_1 = c' + 2$, the factor ‘2’ being the dual Coxeter number.

We also write down the formula which gives the Virasoro central charge through the parameters of the $A_1^{(1)}$ representation in (4.46) (with $\eta = 1$):

$$c = \frac{(1 + 24d)(2 + c') - 6m_1^2}{2 + c'}. \quad (4.53)$$

This formula seems analogous to the one derived in the realization of the Virasoro algebra elements as bilinear combinations (normally ordered) of elements of Kac–Moody algebras which in the $A_1^{(1)}$ case takes the form [243, 392]:

$$c = \frac{3c'}{c' + 2}. \quad (4.54)$$

However, in (4.54) $c' \in \mathbb{N}$ and thus $1 \leq c \leq 3$, while (4.53) is valid for $c \leq 1$. Thus, the only contact we can make between the two formulae is for $c' = c = 1$.

In this case the irreducible Virasoro field has dimension [392] $h = 3/4(2 + c') = 1/4$, which corresponds to $V_0^{h,1}$, $\varepsilon = 1$; cf. (3.42).

It is made irreducible implementing a second order equation obtained from the degree -2 singular vector:

$$v_2 = \left(L_2 - \frac{3}{2(2h+1)} L_1^2 \right) v \Big|_{h=\frac{1}{4}} = (L_2 - L_1^2) v. \quad (4.55)$$

Now in our scheme $2 + c' = m_0 + m_1$, which is consistent with (4.46) with $m_0 = 0$, $c' = 1$, $m_1 = \eta = 3$. Thus the $SU(2)$ representation involved is three-dimensional and this is of course the current J^a , $a = 1, 2, 3$, used in [392].

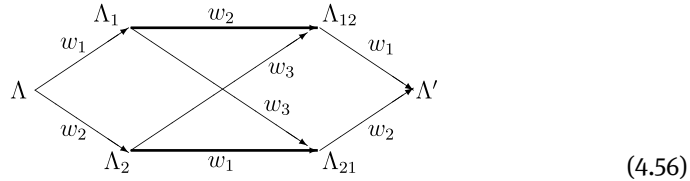
However, there are still open problems in the Virasoro– $A_1^{(1)}$ correspondence. One should try to find a more explicit connection between the corresponding singular vectors (actually two-term compositions) besides the relation between the homogeneity degrees obtained here. The point is that there are some indications that it would be possible to find the general formula for the singular vectors in the Virasoro case using the known $A_1^{(1)}$ singular vectors. (We recall that only some series of Virasoro singular vectors are known; cf. Chapter 3.)

4.1.6 The $A_2^{(1)}$ case

According to Theorem 1 we have nine nonsingular types: $p = \ell_1 = 1, 2, 3$; $0 \leq \nu \leq \delta_{p3}$; $1 \leq p_1 = \ell_{11} \leq p$; $r = q_1 = 0$ in all cases.

The type for $p = 1$ corresponds to the same type for $A_1^{(1)}$ (cf. Remark 1).

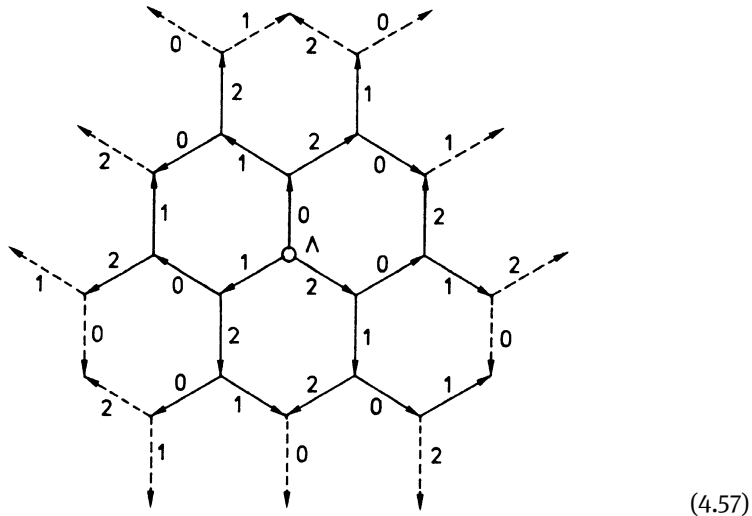
The type for $p = p_1 = 2$ is the sextet diagram for $\mathfrak{sl}(3, \mathbb{C})$:



where $w_3 = w_1 w_2 w_1 = w_2 w_1 w_2$ corresponds to the root $\alpha_3 \equiv \alpha_1 + \alpha_2$, $\Lambda_i = \Lambda - m_i \alpha_i$, $\Lambda_{ij} = \Lambda - m_i \alpha_i - m_j \alpha_j$, $\Lambda' = \Lambda - m_3 \alpha_3$, $m_3 \equiv m_1 + m_2$. (See Volume 1 for details.)

The type for $p = 2$, $p_1 = 1$ is the reduction of (4.56) by setting $m_2 = 0$ (or equivalently $m_1 = 0$). The Verma modules coincide two by two and the corresponding diagram is obtained from (4.56) (as before) by contracting coinciding Verma modules; thus the maps between the latter disappear and the other maps also coincide two by two.

The multiplets of type $p = p_1 = 3$, $\nu = 0$, are represented by the diagram below (cf. [154]).



The honeycomb structure fills the whole plain and each vertex represents a Verma module. There is a central vertex $\Lambda (= \Lambda^*$ of Theorem 2) with signature $\chi = [d, m_0, m_1, m_2]$, $m_i \in \mathbb{N}$, $d \in \mathbb{C}$. From that vertex the three arrows go out. The structure consists of three kinds of sextets: the one depicted in (4.57) and two obtained from it by

replacing w_1, α_1 , or w_2, α_2 with w_0, α_0 (naturally with different Λ). (To avoid cluttering of the diagram we use 0, 1, 2 instead of w_0, w_1, w_2 .) In order to describe the signatures of the other HWMs we first need an explicit representation of the Weyl group W . Let us introduce the following notation involving the simple reflections w_0, w_1, w_2 :

$$a \equiv (xyz) \equiv w_x w_y w_z, \quad x, y, z = 0, 1, 2; \quad x \neq y \neq z \neq x; \\ a^t \equiv (xzy), \quad {}^t a \equiv (yxz), \quad \hat{a} \equiv (zyx); \quad (4.58a)$$

$$\sigma(a) = 0 \quad \text{if } xyz \text{ is an even permutation of } 012; \\ \sigma(a) = 1 \quad \text{if } xyz \text{ is an odd permutation of } 012; \quad (4.58b)$$

$$a_j \equiv (w_x w_y w_z)^j, \quad a'_j \equiv w_z (w_x w_y w_z)^j, \quad a''_j \equiv w_y w_z (w_x w_y w_z)^j; \quad (4.58c)$$

$$W' \equiv \{w_0 w_1 w_2, w_1 w_2 w_0, w_2 w_0 w_1, w_2 w_1 w_0, w_1 w_0 w_2, w_0 w_2 w_1\}. \quad (4.59)$$

Proposition 2. *In terms of the notation in (4.58) and (4.59) the Weyl group W (for $A_2^{(1)}$) is given explicitly by*

$$W = \bigcup_{a \in W'} W_a, \quad (4.60a)$$

$$W_a = \{a_j, a'_j, a''_j; \quad (4.60b)$$

$$a_j^t a_{2k+\sigma(a)}, j \in \mathbb{N}, \quad a_j^t a_{2k+\sigma(a)}, j \in \mathbb{Z}_+, \quad a_j^{t'} a_{2k+\sigma(a)}, j \in \mathbb{Z}_+, \\ k \geq 1 - \sigma(a), \quad (4.60c)$$

$$\hat{a}_j a'_{2k+\sigma(a)}, j \in \mathbb{N}, \quad \hat{a}_j a'_{2k+\sigma(a)}, j \in \mathbb{Z}_+, \quad \hat{a}_j^{t'} a'_{2k+\sigma(a)}, j \in \mathbb{Z}_+, \\ k \geq 1 - \sigma(\hat{a}),$$

$${}^t a_j a''_{2k+\sigma(a)}, j \in \mathbb{N}, \quad {}^t a_j a''_{2k+\sigma(a)}, j \in \mathbb{Z}_+, \quad {}^t a_j^{t'} a''_{2k+\sigma(a)}, j \in \mathbb{Z}_+, \\ k \in \mathbb{Z}_+ \}. \quad \diamond$$

The proof of the proposition is a rather tedious play with the basic properties of the simple reflections,

$$w_x^2 = 1, \quad (w_x w_y)^3 = 1, \quad x, y = 0, 1, 2. \quad (4.61)$$

It is more instructive to inspect the correspondence between W and the honeycomb in the figure. The backbone of the structure are the six infinite chains of Verma modules, to be called main chains, described by (4.60b). Each main chain starts with the central vertex and at every vertex bends in the “other direction” with respect to the previous bending. The three rows of formula (4.60c) describe the vertices between two main chains and to the right (with respect to the direction from Λ to infinity) of the main chain for that a .

We shall give the explicit formulae for the signatures of the Verma modules corresponding to W_a for $a = (012)$ only for the last two “ m_1 ” and “ m_2 ” entries (recall that

the sum of the last three entries is constant and equal to $m_0 + m_1 + m_2$ (m_i of Λ) for every Verma module of the multiplet):

$$\chi(a_j \Lambda) = \begin{cases} m_0 + m_2, m_1 + m_2 + (3j-1)\frac{\tilde{m}}{2}; & j \in 2\mathbb{N} - 1, \\ m_1, m_2 + 3j\frac{\tilde{m}}{2}; & j \in 2\mathbb{Z}_+, \end{cases} \quad (4.62a)$$

$$\chi(a'_j \Lambda) = \begin{cases} m_2 + (3j+1)\frac{\tilde{m}}{2}, -m_1 - m_2 - (3j-1)\frac{\tilde{m}}{2}; & j \in 2\mathbb{N} - 1, \\ m_1 + m_2 + 3j\frac{\tilde{m}}{2}, -m_2 - 3j\frac{\tilde{m}}{2}; & j \in 2\mathbb{Z}_+, \end{cases} \quad (4.62b)$$

$$\chi(a''_j \Lambda) = \begin{cases} -m_2 - (3j+1)\frac{\tilde{m}}{2}, m_0 + m_2; & j \in 2\mathbb{N} - 1, \\ -m_1 - m_2 - 3j\frac{\tilde{m}}{2}, m_1; & j \in 2\mathbb{Z}_+, \end{cases} \quad (4.62c)$$

$$\chi(a_j^t a_{2k} \Lambda) = \begin{cases} m_{12} + (6k+3j-1)\frac{\tilde{m}}{2}, -m_2 - (3k-1)\tilde{m}; & j \in 2\mathbb{N} - 1, \\ m_1 + 3j\frac{\tilde{m}}{2}, m_2 + 3k\tilde{m}; & j \in 2\mathbb{N}, \end{cases} \quad (4.62d)$$

$$\chi(a_j'^t a_{2k} \Lambda) = \begin{cases} -m_{12} - (6k+3j-1)\frac{\tilde{m}}{2}, m_1 + (3j+1)\frac{\tilde{m}}{2}; & j \in 2\mathbb{N} - 1, \\ -m_1 - 3j\frac{\tilde{m}}{2}, m_{12} + 3(2k+j)\frac{\tilde{m}}{2}; & j \in 2\mathbb{Z}_+, \end{cases} \quad (4.62e)$$

$$\chi(a_j''^t a_{2k} \Lambda) = \begin{cases} -m_2 - (3k-1)\tilde{m}, -m_1 - (3j+1)\frac{\tilde{m}}{2}; & j \in 2\mathbb{N} - 1, \\ m_2 + 3k\tilde{m}, -m_{12} - 3(2k+j)\frac{\tilde{m}}{2}; & j \in 2\mathbb{Z}_+, \end{cases} \quad (4.62f)$$

$$\chi(\hat{a}_j a'_{2k+1} \Lambda) = \begin{cases} -m_2 - (3k+1)\tilde{m}, -m_1 - (3j-1)\frac{\tilde{m}}{2}; & j \in 2\mathbb{N} - 1, \\ m_2 + (3k+2)\tilde{m}, -m_{12} - (3k+1+\frac{3}{2}j)\tilde{m}; & j \in 2\mathbb{N}, \end{cases} \quad (4.62g)$$

$$\chi(\hat{a}_j' a'_{2k+1} \Lambda) = \begin{cases} m_1 + (3j+1)\frac{\tilde{m}}{2}, m_2 + (3k+2)\tilde{m}; & j \in 2\mathbb{N} - 1, \\ m_{12} + (3k+2+\frac{3}{2}j)\tilde{m}, -m_2 - (3k+1)\tilde{m}; & j \in 2\mathbb{Z}_+, \end{cases} \quad (4.62h)$$

$$\chi(\hat{a}_j'' a'_{2k+1} \Lambda) = \begin{cases} -m_1 - (3j+1)\frac{\tilde{m}}{2}, m_{12} + (3k+2+\frac{3j+1}{2})\tilde{m}; & j \in 2\mathbb{N} - 1, \\ -m_{12} - (3k+2+\frac{3}{2}j)\tilde{m}, m_1 + (1+\frac{3}{2}j)\tilde{m}; & j \in 2\mathbb{Z}_+, \end{cases} \quad (4.62i)$$

$$\chi({}^t a_j a''_{2k} \Lambda) = \begin{cases} -m_1 - (3j-1)\frac{\tilde{m}}{2}, m_{12} + (3k+\frac{3j+1}{2})\tilde{m}; & j \in 2\mathbb{N} - 1, \\ -m_{12} - 3(2k+j)\frac{\tilde{m}}{2}, m_1 + 3j\frac{\tilde{m}}{2}; & j \in 2\mathbb{N}, \end{cases} \quad (4.62j)$$

$$\chi({}^t a_j' a''_{2k} \Lambda) = \begin{cases} m_2 + (3k+1)\tilde{m}, -m_{12} + (6k+3j+1)\frac{\tilde{m}}{2}; & j \in 2\mathbb{N} - 1, \\ -m_2 - 3k\tilde{m}, -m_1 - 3j\frac{\tilde{m}}{2}; & j \in 2\mathbb{Z}_+, \end{cases} \quad (4.62k)$$

$$\chi({}^t a_j'' a''_{2k} \Lambda) = \begin{cases} m_{12} + (3k+1+\frac{3j+1}{2})\tilde{m}, -m_2 - 3k\tilde{m}; & j \in 2\mathbb{N} - 1, \\ m_1 + (3j+2)\frac{\tilde{m}}{2}, m_2 + (3k+1)\tilde{m}; & j \in 2\mathbb{Z}_+, \end{cases} \quad (4.62l)$$

where $\tilde{m} \equiv m_0 + m_1 + m_2$, $m_{12} \equiv m_1 + m_2$, $k \in \mathbb{N}$ in (4.62d,e,f), $k \in \mathbb{Z}_+$ in (4.62g,h,i,j,k,l).

It is clear that one may introduce other equivalent parametrizations of W . Thus the elements of W corresponding to the positive real roots are most conveniently described as in the following proposition.

Proposition 3. For every positive real root $\beta_{nki} \equiv \beta_{n\kappa\alpha_i}$, $i = 1, 2, 3$; ($\alpha_3 = \alpha_1 + \alpha_2$) of the root system of $A_2^{(1)}$ the corresponding unique element w_{ni}^K of W is given by

$$\begin{aligned} w_{n1}^+ &= w_1(w_0w_2w_0w_1)^n, & w_{n2}^+ &= w_2(w_0w_1w_0w_2)^n, \\ w_{n3}^+ &= w_1w_2w_1(w_0w_1w_2w_1)^n, & n &\in \mathbb{Z}_+, \\ w_{n1}^- &= w_0w_2w_0(w_1w_0w_2w_0)^{n-1}, & w_{n2}^- &= w_0w_1w_0(w_2w_0w_1w_0)^{n-1}, \\ w_{n3}^- &= w_0(w_1w_2w_1w_0)^{n-1}, & n &\in \mathbb{N}. \end{aligned} \quad \diamond$$

The type for $p = 3, p_1 = 2, v = 0$ is obtained from the previous one by setting $m_2 = 0$ (or equivalently $m_0 = 0$ or $m_1 = 0$). Then the multiplets of this type are represented by “half” the honeycomb in the figure. Explicitly, delete the w_2 arrow from the central vertex, then delete the two arrows parallel to it in the two adjacent hexagons, then the two arrows parallel to the previous in the next two adjacent hexagons, etc. Now the structure is divided in two parts which are mirror images; moreover, every Verma module coincides with its mirror image (for $m_2 = 0$); e. g. the deleted arrows were connecting Verma modules Λ' and $\Lambda' - m_2\beta$. Thus, the multiplets are represented by any of the two parts and for future reference we shall keep the one containing the central vertex.

The type for $p = 3, p_1 = 1, v = 0$, is obtained from the above by setting $m_1 = m_2 = 0$ (or equivalently any two of m_0, m_1, m_2). Then the multiplets of this type are shown by “one third” of the previous one. Explicitly also delete the w_1 arrow from the central vertex and so on, as above. The remaining structure is divided into two uneven parts, however, the one containing the central vertex contains all noncoinciding Verma modules exactly once. The structure corresponds to $W_a \cup W_{\tilde{a}}'$, where $a = (210)$, $\tilde{a} = (120)$, $W_{\tilde{a}}'$ is a main chain given by (4.60b).

The types for $p = 3, p_1 = 1, 2, 3, v = 1$ are obtained from those with $v = 0$ by changing the signs of m_i and turning all arrows in the opposite directions.

According to Theorem 3, we have four singular types of multiplets. Two coincide with the ones for $A_1^{(1)}$ described above. The other two are the direct products of the hexagon in (4.56) with Q^d . What is more important is that analogously to the $A_1^{(1)}$ case the latter types may also be represented by the nonsingular types for $p = 3, p_1 = 3, 2$ extended by some finite chains of imaginary reflections of length $m_1 + m_2 - 1$ (for $p_1 = 3$) or $m_1 - 1$ (for $p_1 = 2, m_2 = 0$). Note that all arrows of the honeycomb shall point in one general direction; for m_1, m_2 positive: downwards.

4.2 New Weyl groups for $A_1^{(1)}$ and characters of singular highest weight modules

We consider singular Verma modules over $A_1^{(1)}$, i. e. Verma modules for which the central charge is equal to minus the dual Coxeter number. We calculate the characters of certain factor modules of these Verma modules. In one class of cases we are able

to prove that these factor modules are actually the irreducible highest modules for those highest weights. We introduce new Weyl groups which are infinitely generated abelian groups and are proper subgroups or isomorphic between themselves. Using these Weyl groups we can rewrite the character formulae obtained in this section in the form of the classical Weyl character formula for the finite-dimensional irreducible representations of semisimple Lie algebras (respectively, Weyl–Kac character formula for the integrable highest weight modules over affine Kac–Moody algebras) so that the new Weyl groups play the role of the usual Weyl group (respectively, affine Weyl group).

4.2.1 Introduction

The notion of a Weyl group is very essential for the representation theory of semisimple Lie algebras and groups. It leads to the nice classical formula of Weyl for the characters of the finite-dimensional irreducible representations L of the semisimple Lie algebras. Connectedly it permutes the weights of the finite-dimensional irreducible representations L of the semisimple Lie algebras and determines the embedding pattern of reducible Verma modules over such algebras. Later the notion of a Weyl group was generalized for affine Kac–Moody algebras [354] and for finite-dimensional Lie superalgebras [351]. For affine Kac–Moody algebras the Weyl character formula holds for the integrable highest weight modules L by replacing the Weyl group with the affine Weyl group [349]. The affine Weyl group or Weyl–Kac group determines the embedding pattern of the Verma modules [355, 357], except for the so-called *singular Verma modules*. The latter were introduced in [154] (see Section 4.1) by the property that the central charge is equal to minus the dual Coxeter number or, equivalently, that they are reducible with respect to every imaginary root (see below for definitions).

In [154, 159] in order to describe embeddings between singular Verma modules we introduced reflections (or translations) corresponding to imaginary roots with nontrivial action on the highest weights of the singular Verma modules. In the present section we make the next step, i. e. we introduce a new Weyl group, denoted by W_a , and several of its extensions.

Our criterion is the following. The new Weyl group should be such that the formula for the characters of the analogues of L should look the same as for L by replacing the Weyl or Weyl–Kac group by W_a . Thus we have first to calculate the appropriate characters.

We use results of Malikov–Feigin–Fuchs [436] for the singular vectors of singular Verma modules to calculate the characters of the highest weight modules $F_\Lambda = \mathcal{M}(\Lambda)/I(\Lambda)$, where $\mathcal{M}(\Lambda)$ is a singular Verma module and $I(\Lambda)$ is the submodule of $\mathcal{M}(\Lambda)$ generated by these singular vectors. First we consider singular Verma modules $\mathcal{M}(\Lambda_0)$ which are irreducible with respect to real roots. In this case our result for the character of F_{Λ_0} , $\text{ch } F_{\Lambda_0}$ coincides with $\text{ch } V(\Lambda_0)$, where $V(\Lambda_0)$ is a highest weight

module (HWM) constructed by Wakimoto [566]; thus $F_{\Lambda_0} \cong V(\Lambda_0)$. Then we use the result by Rao [502] that $V(\Lambda_0) \cong L(\Lambda_0)$, the irreducible HWM with weight Λ_0 , thus $F_{\Lambda_0} \cong L(\Lambda_0)$. We further calculate $\text{ch } F_{\Lambda_0^\pm}$ when the $\mathcal{M}(\Lambda_0^\pm)$ are reducible also with respect to some real roots.

4.2.2 Singular Verma modules over $A_1^{(1)}$

Let $\mathcal{G}$ be the affine Lie algebra $A_1^{(1)}$. We return to the settings of Sections 4.1.2 and 4.1.4. Here the only difference would be that here we shall enumerate the simple roots of the $A_1^{(1)}$ as α_1, α_2 instead of α_0, α_1 —this change is because we follow [163] and [436].

Thus, we recall that any $\beta \in \Delta^+$ can be written as in (4.6):

$$\beta = p\alpha_1 + (p + \delta)\alpha_2 \quad p \in \mathbb{Z}_+, \delta \in \{0, \pm 1\}, 2p + \delta > 0, \quad (4.64)$$

where α_1, α_2 are the $A_1^{(1)}$ simple roots.

Furthermore, in the present setting, let $\Lambda_1, \Lambda_2 \in \mathcal{H}^*$ be such that $(\Lambda_i, \alpha_j^\vee) = \delta_{ij}$, (thus $\rho = \Lambda_1 + \Lambda_2$). Then any weight $\Lambda \in \mathcal{H}^*$ can be written as

$$\Lambda = (m_1 - 1)\Lambda_1 + (m_2 - 1)\Lambda_2. \quad (4.65)$$

Thus for a singular Verma module (cf. Section 4.1.3) we have

$$\Lambda = (m_0 - 1)\Lambda_1 - (m_0 + 1)\Lambda_2, \quad m_0 = m_1 = -m_2. \quad (4.66)$$

For $A_1^{(1)}$ in [436] there are exhaustive results for the singular vectors of singular Verma modules. It is shown that the number of singular vectors for fixed $\beta = m\bar{d}$ is equal to the number of partitions of m and all singular vectors are given explicitly. To formulate these results of [436] one needs some notation.

Let $f_3 = [f_1, f_2], f_4 = -[f_1, f_3], f_5 = [f_2, f_3], \dots, f_{3k} = [f_1, f_{3k-1}], f_{3k+1} = -[f_1, f_{3k}], f_{3k+2} = [f_2, f_{3k}]$. The f_i form a basis of $\mathcal{G}_-$, $[f_i, f_j] = \alpha_{ij}f_{i+j}$, where $\alpha_{ij} = -1, 0, 1, \alpha_{ij} \equiv (j - i) \pmod{3}$. One also has $[h_1, f_j] = -2\alpha_0 f_j, [h_2, f_j] = 2\alpha_0 f_j, [e_1, f_j] = 2\alpha_{-1} f_{j-1}, [e_2, f_j] = 2\alpha_{-2} f_{j-2}$.

The singular vectors are given by the following.

Theorem 5 ([436]). *Let $\mathcal{M}(\Lambda)$ be reducible w. r. t. the imaginary roots, i. e. $\Lambda = (m_0 - 1)\Lambda_1 - (m_0 + 1)\Lambda_2$, (cf. (4.66)). Let*

$$F_k = f_1 f_{3k-1} + f_2 f_{3k-2} + \dots + f_{3k-1} f_1 - m_0 f_{3k}, \quad k \in \mathbb{N}. \quad (4.67)$$

For all $k_1, \dots, k_r \in \mathbb{N}$ the vector $F_{k_1} \dots F_{k_r} v_0$ is a singular vector of degree $k = k_1 + \dots + k_r$ of $\mathcal{M}(\Lambda)$. This vector does not depend on the permutation of the numbers $k_1, \dots, k_r$. The vectors $F_{k_1} \dots F_{k_r} v_0$ are linearly independent. $\diamond$

Corollary 2. $F_k F_n = F_n F_k, \forall n, k.$ $\diamond$

Remark 5. The simplest singular vector $F_1 v_0 = (f_1 f_2 + f_2 f_1 - m_0 f_3) v_0 = [(1 - m_0) f_1 f_2 + (m_0 + 1) f_2 f_1] v_0$ was given earlier in [154]; cf. here (4.43). $\diamond$

Let us denote by I_Λ the submodule of $\mathcal{M}(\Lambda)$ generated by the singular vectors of Theorem 5, and by F_Λ the factor module of $\mathcal{M}(\Lambda)$ by I_Λ

$$F_\Lambda = \mathcal{M}(\Lambda)/I_\Lambda. \quad (4.68)$$

Malikov, Feigin and Fuchs [436] made the hypothesis that F_Λ is irreducible if $\mathcal{M}(\Lambda)$ is not reducible w. r. t. the real roots. We shall prove this fact in the next section by calculating first the character of F_Λ , then using a construction of Wakimoto and a result of Rao.

4.2.3 Calculation of characters

Our results will rely on some calculations of characters, so we recall the basic facts following Kac [354].

Let Γ (respectively, Γ_+) be the set of all integer (respectively, integer dominant) elements of $\mathcal{H}^*$, i. e. $\lambda \in \mathcal{H}^*$ such that $(\lambda, \alpha_i) \in \mathbb{Z}$ (respectively, $\mathbb{Z}_+$), $i = 1, 2$.

We recall that for each invariant subspace $V \subset \mathcal{U}(\mathcal{G}_-) v_0 \cong \mathcal{M}(\Lambda)$ we have the following decomposition:

$$V = \bigoplus_{\mu \in \Gamma_+} V^\mu, \quad (4.69)$$

$$V^\mu = \{u \in V \mid h_k \cdot u = (\Lambda - \mu)(h_k)u, k = 1, 2\}. \quad (4.70)$$

(Note that $V^0 = \mathbb{C}v_0$.) Following [354] let $E(\mathcal{H}^*)$ be the associative abelian algebra consisting of the series $\sum_{\mu \in \mathcal{H}^*} c_\mu e(\mu)$, where $c_\mu \in \mathbb{C}$, $c_\mu = 0$ for μ outside the union of a finite number of sets of the form $D(\lambda) = \{\mu \in \mathcal{H}^* \mid \mu \leq \lambda\}$, using any ordering of $\mathcal{H}^*$; the formal exponents $e(\mu)$ have the properties $e(0) = 1$, $e(\mu)e(\nu) = e(\mu + \nu)$. Then we define

$$\begin{aligned} \text{ch } V &= \sum_{\mu \in \Gamma_+} (\dim V^\mu) e(\Lambda - \mu) \\ &= e(\Lambda) \sum_{\mu \in \Gamma_+} (\dim V^\mu) e(-\mu). \end{aligned} \quad (4.71)$$

For the Verma module $\mathcal{M}(\Lambda)$, $\dim V^\mu = P(\mu)$, [349, 354] where $P(\mu)$ is defined as the number of partitions of $\mu \in \Gamma_+$ into a sum of positive roots, where each root is counted with its multiplicity; $P(0) = 1$. We recall several ways to write $\text{ch } \mathcal{M}(\Lambda)$ [349, 354]:

$$\text{ch } \mathcal{M}(\Lambda) = e(\Lambda) \sum_{\mu \in \Gamma_+} P(\mu) e(-\mu) = e(\Lambda) \prod_{\alpha \in \Delta^+} (1 - e(-\alpha))^{-\text{mult } \alpha} \quad (4.72)$$

or, more concretely, for the $A_1^{(1)}$ case setting $e(-\bar{d}) \equiv q$, $e(-\alpha_1) \equiv z$ (then $e(-\alpha_2) = qz^{-1}$), and noting that $\text{mult } \alpha = 1, \forall \alpha \in \Delta^+$:

$$\text{ch } \mathcal{M}(\Lambda) = e(\Lambda) / \prod_{n=1}^{\infty} (1 - q^n)(1 - q^n z^{-1})(1 - q^{n-1} z). \quad (4.73)$$

Let us set

$$V_{j_1 \dots j_m} \equiv \mathcal{U}(\mathcal{G}_-) F_{j_1} \dots F_{j_m} v_0, \quad (4.74)$$

where $F_{j_1} \dots F_{j_m} v_0$ is a singular vector as in Theorem 5. Clearly $V_{j_1 \dots j_m}$ is symmetric in its indices and $V_{j_1 \dots j_m} \subset V_{k_1 \dots k_n}$ if

$$\{k_1, \dots, k_n\} \subset \{j_1, \dots, j_m\} \quad (4.75)$$

the inclusion being proper unless $m = n$. Furthermore, we have

$$\text{ch } V_{j_1 \dots j_m} = q^{j_1 + \dots + j_m} \text{ch } \mathcal{M}(\Lambda_0), \quad \Lambda_0 = (m_0 - 1)\Lambda_1 - (m_0 + 1)\Lambda_1. \quad (4.76)$$

Following [354] we introduce in the ring $\mathbb{C}[(q, z)]$ of formal power series in q, z a partial ordering by putting $f = \sum f_\lambda e(\lambda) \leq g = \sum g_\lambda e(\lambda)$ iff $f_\lambda \leq g_\lambda$ for every λ . Then because of (4.75), (4.76) we have

$$\text{ch } V_{j_1 \dots j_m} < \text{ch } V_{k_1 \dots k_n}, \quad \text{if } \{k_1, \dots, k_n\} \subset \{j_1, \dots, j_m\} \text{ and } n < m. \quad (4.77)$$

The first crucial fact is that we can compute the character of I_{Λ_0} in terms of the characters of $V_{j_1 \dots j_n}$.

Proposition 4. *The following formula holds:*

$$\text{ch } I_{\Lambda_0} = \sum_{n=1}^{\infty} (-1)^{n+1} \sum_{\substack{j_1, \dots, j_n \in \mathbb{N} \\ j_1 < j_2 < \dots < j_n}} \text{ch } V_{j_1 \dots j_n}, \quad \Lambda_0 = (m_0 - 1)\Lambda_1 - (m_0 + 1)\Lambda_2. \quad (4.78)$$

Proof. We shall obtain the formula by an induction procedure.

First we note that the first term ($n = 1$) in (4.78), i. e. $\sum_{j_1 \in \mathbb{N}} \text{ch } V_{j_1}$, should be present in order to account for every possible element of I_{Λ_0} . However, it is clear that

$$\text{ch } I_{\Lambda_0} < \sum_{j_1 \in \mathbb{Z}} \text{ch } V_{j_1} \quad (4.79)$$

because of the overlaps between the V_{j_1} . In particular, each $V_{j_1 j_2}$ ($j_1 < j_2$) is contained in both V_{j_1} and V_{j_2} . This means that the sum $\sum_{j_1, j_2 \in \mathbb{N}, j_1 < j_2} \text{ch } V_{j_1 j_2}$ is contained twice in the RHS of (4.79). Thus, it should be subtracted in order to avoid this overcounting; this is the term with $n = 2$ in (4.78). (Analogously, $V_{j_1 j_2 j_3}$ is contained in $V_{j_1}, V_{j_2}, V_{j_3}, V_{j_1 j_2}, V_{j_1 j_3}, V_{j_2 j_3}$, thus $\text{ch } V_{j_1 j_2 j_3}$ is not contained in the two terms $n = 1, 2$ of (4.78). That is why

it should be added, which gives the third term in (4.78).). Furthermore, we proceed by induction. Suppose the RHS of (4.78) correctly describes the contribution of all $V_{j_1 \dots j_n}$ for $n \leq p-1$, $p > 2$. Consider now a term $V_{j_1 \dots j_p}$. It is contained in $V_{j_1}, \dots, V_{j_p}$, i. e. in $p = \binom{p}{1}$ terms; in $V_{j_1 j_2}, V_{j_1 j_3}, \dots, V_{j_{p-1} j_p}$, i. e. $\binom{p}{2}$ terms; and more generally in $\binom{p}{k}$ terms $V_{j_{i_1} \dots j_{i_k}}$, $k < p$, $\{j_{i_1}, \dots, j_{i_k}\} \subset \{j_1, \dots, j_p\}$. Thus it is contained in $2^p - 2 = \sum_{k=1}^{p-1} \binom{p}{k}$ terms. Next note that the terms with k odd are with sign plus, in (4.78), while those with k even are with sign minus. To calculate how many there are from each sign we note that in $-(a-b)^p = \sum_{k=0}^p \binom{p}{k} (-1)^{k+1} a^{p-k} b^k$, where $a, b > 0$, the terms with plus and minus sign are equal in number. (Indeed, $-(1-1)^p = \sum_{k=0}^p \binom{p}{k} (-1)^{k+1} = 0$.)

The subsum $\sum_{n=1}^{p-1}$ in (4.78) corresponds to $-(a-b)^p$ without the two terms $k=0, p$ ($\binom{p}{0} = \binom{p}{p} = 1$). Thus if p is odd the two missing terms are with opposite signs, and if p is even the two missing terms are with minus sign. Thus if p is odd $V_{j_1 \dots j_p}$ is contained in $2^{p-1} - 1$ terms for which the character enters (4.78) with plus sign and $2^{p-1} - 1$ term for which the character enters (4.78) with minus sign. Thus $V_{j_1 \dots j_p}$ is not taken into account in the first $p-1$ terms of (4.78); thus $\text{ch } V_{j_1 \dots j_p}$ should be *added* with sign plus. Consider now p even, then $V_{j_1 \dots j_p}$ is contained in 2^{p-1} terms for which the character enters (4.78) with sign plus, and $2^{p-1} - 2$ terms for which the character enters (4.78) with sign minus. Thus the net result is that $V_{j_1 \dots j_p}$ is accounted for *twice* and $\text{ch } V_{j_1 \dots j_p}$ should be *subtracted*. So the RHS of (4.82) describes correctly all terms with $n = p$ if the terms of with $n < p$ are described correctly. $\square$

Proposition 5. *The characters of the submodule I_{Λ_0} , $\Lambda_0 = (m_0 - 1)\Lambda_1 - (m_0 + 1)\Lambda_2$ and of the factor module $F_{\Lambda_0} \equiv \mathcal{M}(\Lambda_0)/I_{\Lambda_0}$ are given by*

$$\text{ch } I_{\Lambda_0} = \text{ch } \mathcal{M}(\Lambda_0) \left[1 - \prod_{n=1}^{\infty} (1 - q^n) \right], \quad (4.80)$$

$$\begin{aligned} \text{ch } F_{\Lambda_0} &= \text{ch } \mathcal{M}(\Lambda_0) \prod_{n=1}^{\infty} (1 - q^n) = e(\Lambda_0) / \prod_{n=1}^{\infty} (1 - q^{n-1}z)(1 - q^n z^{-1}) \\ &= e(\Lambda_0) / \prod_{\alpha \in \Delta_R^+} (1 - e(-\alpha)), \end{aligned} \quad (4.81)$$

where Δ_R^+ is the set of positive real roots.

Proof. We have

$$\text{ch } F_{\Lambda_0} = \text{ch } \mathcal{M}(\Lambda_0) - \text{ch } I_{\Lambda_0}, \quad (4.82)$$

thus (4.81) follows from (4.80), (4.72) and (4.73). For $\text{ch } I_{\Lambda_0}$ we have, substituting (4.76) in (4.78),

$$\text{ch } I_{\Lambda_0} = \text{ch } \mathcal{M}(\Lambda_0) \sum_{n=1}^{\infty} (-1)^{n+1} \sum_{\substack{j_1, \dots, j_n \in \mathbb{N} \\ j_1 < \dots < j_n}} q^{j_1 + \dots + j_n} \quad (4.83)$$

$$= \text{ch } \mathcal{M}(\Lambda_0) \left[1 - \sum_{n=0}^{\infty} (-1)^n \sum_{\substack{j_1, \dots, j_n \in \mathbb{N} \\ j_1 < \dots < j_n}} q^{j_1 + \dots + j_n} \right], \quad (4.84)$$

which is equal to (4.80). $\square$

In [566] Wakimoto constructed a family of highest weight modules $\pi_{\mu\nu}$ over $A_1^{(1)}$ parametrized by $(\mu, \nu) \in \mathbb{C}^2$. We shall not repeat the construction of the $\pi_{\mu\nu}$ representation spaces $V(\mu, \nu)$ from [566]. We need only two facts. The highest weight module $V(\mu, 0)$ has weight $\Lambda_{\mu,0} = -(1+\mu)\Lambda_1 + (\mu-1)\Lambda_1$, i. e. the highest weight Λ_0 of a singular Verma module (cf. (1.56) with $\mu = -m_0$). Furthermore it is shown that [566]

$$\text{ch } V(\mu, 0) = e(\Lambda_{\mu,0}) / \prod_{\alpha \in \Delta_k^+} (1 - e(-\alpha)), \quad (4.85)$$

which coincides with (4.81). Consequently

$$F_{\Lambda_0} \cong V(\mu, 0), \quad \Lambda_0 = \Lambda_{\mu,0}. \quad (4.86)$$

Next we use an observation due to Rao [502].

Theorem 6 ([502]). *The modules $V(\mu, 0)$ constructed by Wakimoto are irreducible if $\mu \notin \mathbb{Z} \setminus \{0\}$.* $\diamond$

A necessary condition for the irreducibility of $V(\mu, 0)$ was obtained in [160] (Theorem B). Combining Proposition 5, (4.86) and Theorem 6 we have proved the following.

Theorem 7. *Let $L(\Lambda_0)$ be the irreducible highest weight module with highest weight $\Lambda_0 = (m_0 + 1)\Lambda_1 - (m_0 + 1)\Lambda_2$, $m_0 \notin \mathbb{Z} \setminus \{0\}$, such that the Verma module $\mathcal{M}(\Lambda_0)$ is reducible with respect to the imaginary roots and irreducible with respect to the real roots. Then*

$$L(\Lambda_0) \cong F_{\Lambda_0} \cong \mathcal{M}(\Lambda_0) / I_{\Lambda_0} \cong V(-m_0, 0), \quad (4.87)$$

where I_{Λ_0} is the submodule of $\mathcal{M}(\Lambda_0)$ generated by the singular vectors of Malikov, Feigin, and Fuchs from Theorem 5, and $V(-m_0, 0)$ is the highest weight module with highest weight Λ_0 , constructed by Wakimoto. In particular,

$$\text{ch } L(\Lambda_0) = e(\Lambda_0) / \prod_{\alpha \in \Delta_k^+} (1 - e(-\alpha)) = e(\Lambda_0) / \prod_{k=1}^{\infty} (1 - q^k z^{-1})(1 - q^{k-1} z). \quad (4.88)$$

$\diamond$

For $\Lambda_0 = -\rho = -\Lambda_1 - \Lambda_2$, i. e. $m_0 = 0$, formula (4.88) was conjectured in [355].

Next we consider the case when $\mathcal{M}(\Lambda_0)$ is reducible w. r. t. to the imaginary roots and to some real roots. Consider first the case when $\mathcal{M}(\Lambda_0)$ is singular and reducible also w. r. t. the real roots with $\kappa = -1$ (cf. (4.6)), i. e. (4.15) holds for a fixed $m \in \mathbb{N}$, and $\beta = k\bar{d} + \alpha_1$, $k = 0, 1, \dots$. Then according to (4.66) we have $m_2 = -m$, $\Lambda_0 \rightarrow \Lambda_0^+ = (m-1)\Lambda_1 - (m+1)\Lambda_2$, $m \in \mathbb{N}$. Thus all Verma modules $\mathcal{M}(\Lambda_0^+ - m\beta) = \mathcal{M}(\Lambda_0^+ - m(p-1)\bar{d} -$

$m\alpha_1$), $p \geq 1$, are isomorphic to submodules of $\mathcal{M}(\Lambda_0^+)$. Moreover, all these are singular Verma modules and for $p > 1$ all are isomorphic to submodules of $\mathcal{M}(\Lambda_0^+ - m\alpha_1)$. $\mathcal{M}_n = \mathcal{M}(\Lambda_0^+ - n\bar{d} - \alpha_1)$ are isomorphic to submodules of $\mathcal{M}(\Lambda_0^+ - m\alpha_1)$ and thus to submodules of $\mathcal{M}(\Lambda_0^+)$. The most general statement is the following.

Proposition 6. *Let $\mathcal{M}(\Lambda_0^+)$ be a singular Verma module reducible w. r. t. the positive roots $\beta = p\alpha_1 + (p-1)\alpha_2$, $\Lambda_0^+ = (m-1)\Lambda_1 - (m+1)\Lambda_2$, $m \in \mathbb{N}$. Then we have the following invariant embeddings of Verma modules:*

$$\mathcal{M}(\Lambda_0^+ - n\bar{d} - \varepsilon m\alpha_1) \supset \mathcal{M}(\Lambda_0^+ - n'\bar{d} - \varepsilon' m\alpha_1), \quad (4.89)$$

$$n, n' \in \mathbb{Z}; \quad \varepsilon, \varepsilon' = 0, 1; \quad n \leq n'; \quad n = n' \Rightarrow \varepsilon = 0, \varepsilon' = 1;$$

$$\varepsilon = 1, \varepsilon' = 0 \Rightarrow n' - n \geq m.$$

Proof. First we note that $\mathcal{M}(\Lambda_0^+ - n\bar{d} - \varepsilon m\alpha_1)$ is reducible w. r. t. the imaginary roots for $\forall n \in \mathbb{Z}$, $\varepsilon = 0, 1$; $\mathcal{M}(\Lambda_0^+ - n\bar{d})$ is reducible w. r. t. all positive roots $\beta = p\alpha_1 + (p-1)\alpha_2$, $p \geq 1$, so that (4.15) holds with the same m from Λ_0^+ ; $\mathcal{M}(\Lambda_0^+ - n\bar{d} - m\alpha_1)$ is reducible w. r. t. all positive roots $\beta = (p-1)\alpha_1 + p\alpha_2$, $p \geq 1$, so that (4.15) holds with the same m from Λ_0^+ .

For $\varepsilon = \varepsilon' = 0, 1$ the embeddings in (4.89) are described by the singular vectors of degree $n' - n (> 0)$ of Theorem 5. For $\varepsilon = 0, \varepsilon' = 1$ and $n' - n = km$, $k = 0, 1, \dots$, the embeddings in (4.89) are described by the singular vectors $F(m, p, t)v_0$ of Theorem 3.2 of [436], which are valid also for nonsingular Verma modules ($t \equiv m_1 + m_2$). In our case $t = m_1 + m_2 = 0$ and we have

$$v_s = v_s^{m\beta} = F_1(m, k)v_0, \quad \beta = k\bar{d} + \alpha_1, \quad k = \frac{n' - n}{m} = 0, 1, \dots, \quad (4.90a)$$

$$F_1(m, k) \equiv F(m, k+1, 0) = (f_1^m f_2^m)^k f_1^m. \quad (4.90b)$$

(Note that $F_1(m, 0) = f_1^m$ as it should; cf. (4.35).) For $\varepsilon = 0, \varepsilon' = 1$ and $n' - n \neq km$, $k = 0, 1, \dots$, the embeddings in (4.89) are described by composition of embeddings of the two types described so far. Let $n' - n = km + \ell$, $k = 0, 1$, $\ell = 1, 2, \dots, m-1$; then (4.89) is described, e. g. by

$$v_s = F_{\ell_1} \dots F_{\ell_s} F_1(m, k)v_0, \quad \ell_1 + \dots + \ell_s = \ell, \quad (4.91)$$

F_{ℓ_i} are given by formula (4.67). For $\varepsilon = 1, \varepsilon' = 0$, $n' - n = km$, $k = 1, 2, \dots$ the embeddings are described again by the singular vectors of Theorem 3.2 of [436], however, with f_1, f_2 interchanged, i. e.

$$v_s = F_2(m, k)v_0 = (f_2^m f_1^m)^k f_2^m. \quad (4.92)$$

For $\varepsilon = 1, \varepsilon' = 0$, $n' - n = km + \ell$, $k = 1, 2, \dots$, $\ell = 1, 2, \dots, m-1$ the embeddings are described, e. g. by

$$v_s = F_{\ell_1} \dots F_{\ell_s} F_2(m, k)v_0, \quad \ell_1 + \dots + \ell_s = \ell. \quad (4.93)$$

□

It is clear that the structure of $\mathcal{M}(\Lambda_0^+)$ is determined only by the non-composition embeddings described in Proposition 6. Some further compositions are obvious, i. e. from (4.89) we see

$$\mathcal{M}(\Lambda_0^+) \supset \mathcal{M}(\Lambda_0^+ - m\alpha_1) \supset \cdots \supset \mathcal{M}(\Lambda_0^+ - k\bar{d} - m\alpha_1) \supset \cdots \quad (4.94)$$

and, moreover, the singular vectors in (4.90a) have the property

$$v_s = F_1(m, k)v_0 = (f_1^m f_2^m)^k F_1(m, 0)v_0. \quad (4.95)$$

Thus we have proved the following.

Proposition 7. *Let $\mathcal{M}(\Lambda_0^+)$ be a singular Verma module reducible w. r. t. the positive roots $\beta = p\alpha_1 + (p-1)\alpha_2$, $p \geq 1$, $\Lambda_0^+ = (m-1)\Lambda_1 - (m+1)\Lambda_2$, $m \in \mathbb{N}$. Then the only noncomposition embeddings of Verma modules in $\mathcal{M}(\Lambda_0^+)$ are those generated by the singular vectors corresponding to reducibility w. r. t. the imaginary roots and w. r. t. the simple root $\beta = \alpha_1$ ($p = 1$), i. e. $v_s = f_1^m v_0$ (cf. (4.35)).* $\diamond$

Now we can prove the following.

Proposition 8. *Let $\tilde{I}_{\Lambda_0^+}$ be the submodule of $\mathcal{M}(\Lambda_0^+)$ generated by the singular vectors of Theorem 5 and $v_s = f_1^m v_0$.*

Let

$$\tilde{F}_{\Lambda_0^+} = \mathcal{M}(\Lambda_0^+)/\tilde{I}_{\Lambda_0^+}. \quad (4.96)$$

Then

$$\begin{aligned} \text{ch } \tilde{F}_{\Lambda_0^+} &= (1 - z^m) \text{ch } F_{\Lambda_0^+} \\ &= (1 - z^m) e(\Lambda_0^+) / \prod_{n=1}^{\infty} (1 - q^n z^{-1})(1 - q^{n-1} z) \\ &= e(\Lambda_0^+) (1 + z + z^2 + \cdots + z^{m-1}) / \prod_{n=1}^{\infty} (1 - q^n z^{-1})(1 - q^n z), \end{aligned} \quad (4.97)$$

where $\text{ch } F_{\Lambda_0^+}$ is from (4.81).

Proof. Let $V^m = \mathcal{U}(\mathcal{G}_-) f_1^m v_0$, $\tilde{I}_{\Lambda_0^+} = V^m \cup I_{\Lambda_0^+}$, where $I_{\Lambda_0^+}$ is the submodule generated by the singular vectors of Theorem 5. We can also write

$$\tilde{I}_{\Lambda_0^+} = V^m \cup (I_{\Lambda_0^+} \setminus V^m), \quad (4.98)$$

then

$$\text{ch } \tilde{I}_{\Lambda_0^+} = \text{ch } V^m + \text{ch } (I_{\Lambda_0^+} \setminus V^m). \quad (4.99)$$

We use $f_1 \leftrightarrow e(-\alpha_1) = z$ and

$$\text{ch } V^m = z^m \text{ch } \mathcal{M}(\Lambda_0^+), \quad (4.100a)$$

$$\text{ch}(I_{\Lambda_0^+} \setminus V^m) = (1 - z^m) \text{ch } I_{\Lambda_0^+}, \quad (4.100b)$$

to obtain taking into account (4.80) and (4.81)

$$\text{ch } \tilde{F}_{\Lambda_0^+} = \text{ch } \mathcal{M}(\Lambda_0^+) - \text{ch } \tilde{I}_{\Lambda_0^+} = (1 - z^m) \text{ch } F_{\Lambda_0^+}. \quad (4.101)$$

□

Analogously we can consider the singular Verma modules $\mathcal{M}(\Lambda_0^-)$ reducible w. r. t. the real roots $\beta = (p-1)\alpha_1 + p\alpha_2$ ($p \geq 1, \kappa = 1$ in (4.6)). In this case (cf. (4.66)) $\Lambda_0^- = -(m+1)\Lambda_1 + (m-1)\Lambda_2$, $m \in \mathbb{N}$.

Proposition 9. *Let $\mathcal{M}(\Lambda_0^-)$ be a singular Verma module, reducible w. r. t. the positive roots $\beta = (p-1)\alpha_1 + p\alpha_2$, $\Lambda_0^- = -(m+1)\Lambda_1 + (m-1)\Lambda_2$, $m \in \mathbb{N}$. Then the only noncomposition embeddings of Verma modules in $\mathcal{M}(\Lambda_0^-)$ are those generated by the singular vectors corresponding to reducibility w. r. t. the imaginary roots and w. r. t. the simple root $\beta = \alpha_2$, ($p = 1$), i. e. $v_s = f_2^m v_0$ (cf. (4.35)). Let $\tilde{I}_{\Lambda_0^-}$ be the submodule of $\mathcal{M}(\Lambda_0^-)$ generated by the above singular vectors. Let*

$$\tilde{F}_{\Lambda_0^-} = \mathcal{M}(\Lambda_0^-) / \tilde{I}_{\Lambda_0^-}. \quad (4.102)$$

Then

$$\begin{aligned} \text{ch } \tilde{F}_{\Lambda_0^-} &= (1 - q^m z^{-m}) \text{ch } F_{\Lambda_0^-} \\ &= (1 - q^m z^{-m}) e(\Lambda_0^-) / \prod_{n=1}^{\infty} (1 - q^n z^{-1})(1 - q^{n-1} z) \\ &= \frac{e(\Lambda_0^-)(1 + qz^{-1} + q^2 z^{-2} + \dots + q^{m-1} z^{-m+1})}{\prod_{n=1}^{\infty} (1 - q^{n+1} z^{-1})(1 - q^{n-1} z)}. \end{aligned} \quad (4.103)$$

Proof. The proof is analogous to the proof of Propositions 7 and 8. Note that $f_2 \leftrightarrow e(-\alpha_2) = e(-\bar{d} + \alpha_1) = qz^{-1}$. □

Corollary 3. *Let $\tilde{I}_{\Lambda_0^+}^n$ ($\tilde{I}_{\Lambda_0^-}^n$) be the submodule of $\mathcal{M}(\Lambda_0^+ - n\bar{d})$ (respectively, $\mathcal{M}(\Lambda_0^- - n\bar{d})$) generated by the singular vectors of Theorem 5, and $v_s = f_1^m v_0$, (resp $v_s = f_2^m v_0$) and let*

$$\tilde{F}_{\Lambda_0^\pm}^n = \mathcal{M}(\Lambda_0^\pm - n\bar{d}) / \tilde{I}_{\Lambda_0^\pm}^n. \quad (4.104)$$

Then

$$\text{ch } \tilde{F}_{\Lambda_0^+}^n = (1 - z^m) e(\Lambda_0^+ - n\bar{d}) / \prod_{n=1}^{\infty} (1 - q^n z^{-1})(1 - q^{n-1} z), \quad (4.105)$$

$$\text{ch } \tilde{F}_{\Lambda_0^-}^n = (1 - q^m z^{-m}) e(\Lambda_0^- - n\bar{d}) / \prod_{n=1}^{\infty} (1 - q^n z^{-1})(1 - q^{n-1} z). \quad (4.106)$$

4.2.4 Weyl groups for the singular highest weight modules

As we have noted in Section 4.1.2 (and in [154, 159]) the Weyl–Kac group W [354] which is generated by the reflections s_{α_i} , where α_i ($i = 0, \dots, \ell$), are the simple roots for an affine Kac–Moody algebra $\mathcal{G}$, is not adequate for the description of the singular highest weight modules (HWMs). Thus, in [154, 159] and here in (4.19) we introduced the notion of *imaginary reflections* or “*translations*” s_{β} , $\beta = n\bar{d}$ imaginary.

We used these translations to obtain a description of the embedding pattern of submodules of the singular Verma modules in the same way as the Weyl–Kac group describes the embedding pattern of the Verma modules with dominant integral highest weight [355]. Equivalently, these translations describe the possible multiplets of singular Verma modules [154, 159].

Here we develop further these ideas to introduce a Weyl group suitable for the character formula for $L(\Lambda_0)$, Λ_0 singular highest weight.

Consider the infinite abelian multiplicative group W_a generated by the symbols $w(n)$, $n \in \mathbb{N}$, with the properties $w(n)^2 = 1$, $w(n)w(n') = w(n')w(n)$. Such a group is called a torsion group or a p -group [248, 365] (here $p = 2$). The group W_a may be parametrized as follows:

$$W_a = \{w = w(n_1) \dots w(n_k) \mid k \in \mathbb{Z}_+; \text{ for } k = 0, w = 1; \quad (4.107)$$

$$\text{for } k > 0, n_i \in \mathbb{N}, n_1 < \dots < n_k\}.$$

If $w = 1$, or $w = w(n_1) \dots w(n_k)$, $n_i \in \mathbb{N}$, $n_i \neq n_j$, $i \neq j$, we shall say that w is given in a *reduced form*. It is clear that every element of W_a has a reduced form.

We define the action of $w \in W_a$ on $\mathcal{H}^*$ analogously to the action of $s_{n\bar{d}}$ in (4.19). For $\lambda \in \mathfrak{h}^*$ and w in a reduced form we set

$$w \cdot \lambda = w(n_1) \dots w(n_k) \cdot \lambda \equiv \lambda - (n_1 + \dots + n_k)\bar{d}, \quad n_i \in \mathbb{N}, n_i \neq n_j, 1 \cdot \lambda = \lambda. \quad (4.108)$$

This action is non-associative, i. e. it may happen that $(w_1 w_2) \cdot \lambda \neq w_1 \cdot (w_2 \cdot \lambda)$, but this is not essential for our purposes. The action is associative if we consider W_a as a semi-group (i. e. if we drop the relation $w(n)^2 = 1$).

Analogously to the Weyl(–Kac) group we introduce the *length* of $w \in W_a$, denoted $\ell(w)$, for w given in a reduced form:

$$\ell(w) = k, \quad w = w(n_1) \dots w(n_k), \quad n_i \in \mathbb{N}, n_i \neq n_j, i \neq j, \ell(1) = 0. \quad (4.109)$$

The length of an arbitrary element W_a is defined to be equal to the length of its reduced form. (Note that an element of length k has $k!$ reduced forms.) Now we can prove the following.

Proposition 10. *Let $\mathcal{M}(\Lambda_0)$, F_{Λ_0} , and $L(\Lambda_0)$ be as in Proposition 5, $\Lambda_0 = (m-1)\Lambda_1 - (m+1)\Lambda_2$. Let W_a be the Weyl group in (4.107). Then we have*

$$\text{ch } F_{\Lambda_0} = \text{ch } \mathcal{M}(\Lambda_0) \sum_{w \in W_a} (-1)^{\ell(w)} e(w \cdot (\Lambda_0 + \rho) - \Lambda_0 - \rho), \quad (4.110a)$$

$$\begin{aligned} (\Lambda_0 + \rho, \bar{d}) &= 0, \\ \text{ch } F_{\Lambda_0} &= \text{ch } L(\Lambda_0), \quad \text{if } m \notin \mathbb{Z} \setminus \{0\}. \end{aligned} \quad (4.110b)$$

Proof. Consider the sum in the RHS of (4.110a),

$$\sum_{w \in W_a} (-1)^{\ell(w)} e(w \cdot (\Lambda_0 + \rho) - \Lambda_0 - \rho) \quad (4.111a)$$

$$\begin{aligned} &= \sum_{k=0}^{\infty} \sum_{\substack{n_1 < \dots < n_k \\ n_1, \dots, n_k \in \mathbb{N}}} (-1)^{\ell(w(n_1) \dots w(n_k))} \\ &\quad \times e((w(n_1) \dots w(n_k)) \cdot (\Lambda_0 + \rho) - \Lambda_0 - \rho) \end{aligned} \quad (4.111b)$$

$$= \sum_{k=0}^{\infty} (-1)^k \sum_{\substack{n_1, \dots, n_k \in \mathbb{N} \\ n_1 < \dots < n_k}} e(-n_1 \bar{d} - n_2 \bar{d} \dots n_k \bar{d}) \quad (4.111c)$$

$$= \sum_{k=0}^{\infty} (-1)^k \sum_{\substack{n_1, \dots, n_k \in \mathbb{N} \\ n_1 < \dots < n_k}} q^{n_1 + \dots + n_k} \quad (4.111d)$$

$$= \prod_{n=1}^{\infty} (1 - q^n). \quad (4.111e)$$

Comparing with (4.81) and (4.88) we see that the proof is finished. $\square$

Consider now the subgroups S^+ , S^- of the Weyl–Kac group W so that $S^+ = \{1, s_1\}$, $S^- = \{1, s_2\}$ and $s_a = s_{\alpha_a}$ is given in (4.17). Note that each one of S^+ , S^- is isomorphic to the Weyl group W_0 of the $A_1 = \mathfrak{sl}(2, \mathbb{C})$, the finite-dimensional algebra underlying $A_1^{(1)}$. Consider further the infinite abelian groups $W_a^{\pm}$ with generators s_1 or s_2 and the generators of W_a , i. e.

$$W_a^{\pm} = \{w = w's = sw' \mid w' \in W_a, s \in S^{\pm}\}. \quad (4.112)$$

This definition is meaningful since $s_a w(n) \cdot \Lambda = s_a \cdot (\Lambda - n\bar{d}) = \Lambda - n\bar{d} - 2 \frac{(\Lambda - n\bar{d}, \alpha)}{(\alpha, \alpha)} \alpha = \Lambda - \frac{2(\Lambda, \alpha)}{(\alpha, \alpha)} \alpha - n\bar{d} = w(n)s_a \cdot \Lambda$. Thus

$$W_a^{\pm} \cong W_a \times W_0. \quad (4.113)$$

We set

$$\ell(w) = \ell_a(w') + \ell_0(s), \quad w = w's, \quad w' \in W_a, s \in S^{\pm}, \quad \ell_0(s_a) = 1, \quad \ell_0(1) = 0. \quad (4.114)$$

Next we prove the analogue of Proposition 10 for $F_{\Lambda_0^+}$.

Proposition 11. *Let $\mathcal{M}(\Lambda_0^{\pm})$, $\tilde{F}_{\Lambda_0^{\pm}}$ be as in Propositions 6, 7, 8, $\Lambda_0^{\pm} = (\pm m - 1)\Lambda_1 - (\pm m + 1)\Lambda_2$, $m \in \mathbb{N}$. Let $W_a^{\pm}$ be the Weyl groups in (4.112) and (4.113). Then we have*

$$\text{ch } \tilde{F}_{\Lambda_0^{\pm}} = \text{ch } \mathcal{M}(\Lambda_0^{\pm}) \sum_{w \in W_a^{\pm}} (-1)^{\ell(w)} e(w \cdot (\Lambda_0^{\pm} + \rho) - \Lambda_0^{\pm} - \rho). \quad (4.115)$$

Proof. Consider the sum in the RHS of (4.115) for the plus sign:

$$\begin{aligned}
 & \sum_{w \in W_a^+} (-1)^{\ell(w)} e(w \cdot (\Lambda_0^+ + \rho) - \Lambda_0^+ - \rho) \\
 &= \sum_{w \in W_a} (-1)^{\ell(w)} e(w \cdot (\Lambda_0^+ + \rho) - \Lambda_0^+ - \rho) \\
 &\quad - \sum_{w \in W_a} (-1)^{\ell(w)} e(w \cdot (s_1 \cdot (\Lambda_0^+ + \rho) - \Lambda_0^+ - \rho)). \tag{4.116}
 \end{aligned}$$

By the definition of Λ_0^+ , $s_1 \cdot (\Lambda_0^+ + \rho) = \Lambda_0^+ + \rho - m\alpha_1$, i. e. we have

$$\begin{aligned}
 w \cdot (s_1 \cdot (\Lambda_0^+ + \rho)) &= w \cdot (\Lambda_0^+ + \rho - m\alpha_1) \\
 &= w \cdot (\Lambda_0^+ + \rho) - m\alpha_1, \\
 e(w \cdot (s_1 \cdot (\Lambda_0^+ + \rho)) - \Lambda_0^+ - \rho) \\
 &= e(w \cdot (\Lambda_0^+ + \rho) - \Lambda_0^+ - \rho - m\alpha_1) \\
 &= e(w \cdot (\Lambda_0^+ + \rho) - \Lambda_0^+ - \rho) e(-m\alpha_1), \tag{4.117}
 \end{aligned}$$

so the RHS of (4.116) becomes (using also (4.110a))

$$\begin{aligned}
 & (1 - e(-m\alpha_1)) \sum_{w \in W_a} (-1)^{\ell(w)} e(w \cdot (\Lambda_0^+ + \rho) - \Lambda_0^+ - \rho) \\
 &= (1 - z^m) \operatorname{ch} F_{\Lambda_0^+} / \operatorname{ch} \mathcal{M}(\Lambda_0^+), \tag{4.118}
 \end{aligned}$$

which proves (4.115) for the plus sign. For the minus sign, we use the fact that by definition $s_2(\Lambda_0^- + \rho) = \Lambda_0^- + \rho - m\alpha_2$. $\square$

We note that (4.110) and (4.115) have exactly the form of the Weyl (respectively, Weyl–Kac) character formula for the irreducible finite-dimensional representations of semi-simple Lie algebras (respectively, for the integrable representations of affine Kac–Moody algebras); one should replace W_a or $W_a^\pm$ with the Weyl group W_0 (respectively, Weyl–Kac group W). One interesting question is what the relation is between the Weyl–Kac group W and the new Weyl groups $W_a, W_a^\pm$.

The Weyl group W is generated by the two simple reflections s_1, s_2 (cf. (4.17)). In (4.31), (4.32), and (4.33) we have actually used

$$W = \{(s_1 s_2)^k, (s_2 s_1)^\ell, (s_1 s_2)^m s_1, (s_2 s_1)^n s_2 \mid k, \ell, m, n \in \mathbb{Z}_+\}. \tag{4.119}$$

We recall the action of W on the weights $\Lambda \in \mathcal{H}^*$ as given in (4.30) for $\Lambda = \Lambda_{00} = \Lambda_{01} = d\bar{d} + \lambda_1 \alpha_1 + c\bar{c}$, $d, \lambda_1, c \in \mathbb{C}$, where

$$\begin{aligned}
 \chi(\Lambda) &\equiv [d, m_1, m_2], \quad m_i \equiv (\Lambda + \rho, \alpha_i), \quad i = 1, 2, \\
 m_1 &= 2\lambda_1 + 1, \quad m_2 = c - 2\lambda_1 + 1. \tag{4.120}
 \end{aligned}$$

We note for further use the correspondence

$$W_a \ni W(mk) \mapsto (s_2 s_1)^k, \quad k \in \mathbb{Z}_+, \quad (4.121)$$

which although with coinciding action on Λ_0^+ is not an isomorphism even between semigroups because, e. g. $w(m(k_1 + k_2))$ and $w(mk_1)w(mk_2)$ are mapped to $(s_2 s_1)^{k_1 + k_2}$. The same applies for the correspondence

$$W_a^+ \ni sw(mk) \mapsto s_1(s_2 s_1)^k \in W, \quad k \in \mathbb{Z}_+, \quad (4.122)$$

(coinciding on Λ_0^+), because $s_1(s_2, s_1)^k$ do not form a subgroup of W .

In [154] we considered an extended Weyl group W_e which in the language here should be described as an infinite abelian group with generators $w(n)s = sw(n)$, $n \in \mathbb{Z}$, $s \in S^+$ (or S^-), i. e.

$$W_e^\pm = \{w' = ws \mid w = w(n_1) \dots w(n_k), k \in \mathbb{Z}_+, \quad (4.123)$$

$$w = 1 \text{ if } k = 0, n_i \in \mathbb{Z} \setminus \{0\},$$

$$w(n)^2 = 1, w(n)w(n') = w(n')w(n), s \in S^\pm, sw = ws\}.$$

Thus we have the following inclusions of groups:

$$W_a \subset W_a^\pm \subset W_e^\pm. \quad (4.124)$$

If we consider the action of the Weyl–Kac group W on Λ_0^+ , besides (4.121) and (4.122) we have the following correspondences:

$$W \ni (s_1 s_2)^k \mapsto w(-mk) \in W_e^+, \quad k = 0, 1, \dots, \quad (4.125)$$

$$W \ni s_2(s_1 s_2)^k \mapsto sw(-m(k+1)) \in W_e^+, \quad k = 0, 1, \dots, \quad (4.126)$$

however, again these maps give rise to no isomorphism.

Finally, we introduce a notion which will make all formulae for the characters uniform. Let W^* be any of the Weyl groups we considered, let $\Lambda \in \mathcal{H}^*$ and V be a HWM with highest weight Λ . Then we shall say that $w \in W^*$ is *V-active* if there exists a HWM V' with highest weight $w(\Lambda + \rho) - \rho$ which is isomorphic to a submodule of V . Further we shall restrict this definition to the category of Verma modules, i. e. V and V' above should be Verma modules. Thus we have the following.

Proposition 12. *Let Λ_0 be as in Proposition 10, with $m \notin \mathbb{Z} \setminus \{0\}$, $\Lambda_0^\pm$ as in Proposition 11. Then we have*

$$\begin{aligned} \text{ch } L(\Lambda_0) &= \text{ch } \mathcal{M}(\Lambda_0) \sum_{\substack{w \in W_a^\pm \\ w - \mathcal{M}(\Lambda_0)\text{-active}}} (-1)^{\ell(w)} e(w \cdot (\Lambda_0 + \rho) - \Lambda_0 - \rho) \\ &= \text{ch } \mathcal{M}(\Lambda_0) \sum_{\substack{w \in W_e^\pm \\ w - \mathcal{M}(\Lambda_0)\text{-active}}} (-1)^{\ell(w)} e(w(\Lambda_0 + \rho) - \Lambda_0 - \rho), \end{aligned} \quad (4.127)$$

$$\text{ch } \tilde{F}_{\Lambda_0^\pm} = \text{ch } \mathcal{M}(\Lambda_0^\pm) \sum_{\substack{w \in W_e^\pm \\ w - \mathcal{M}(\Lambda_0^\pm)\text{-active}}} (-1)^{\ell(w)} e(w(\Lambda_0^\pm + \rho) - \Lambda_0^\pm - \rho). \quad (4.128)$$

4.2.5 Discussion

Several paths of applications and generalizations are available. First one should look for new modular invariants connected with the characters calculated in this Section. For this we note that all characters (cf. e. g. (4.81) and (4.88)) contain the factor

$$\psi(q, z) = 1 / \prod_{n=1}^{\infty} (1 - q^n z^{-1})(1 - q^{n-1} z) = \frac{q^{1/12} z^{-1/2} \eta(\tau)}{\theta_{1,2}(\zeta, \tau) - \theta_{-1,2}(\zeta, \tau)}, \quad (4.129)$$

where $q = e^{2\pi i \tau}$, $z = e^{2\pi i \zeta}$, $\eta(\tau)$ is the Dedekind function, and $\theta_{n,m}(\zeta, \tau) = \theta_{n,m}(\zeta, \tau, 0)$ where the latter is the classical theta function associated with $A_1^{(1)}$ so that [354]

$$\theta_{n,m}(\zeta, \tau, u) = e^{-2\pi i m u} \sum_{k \in \mathbb{Z} + n/2m} e^{2\pi i k(k^2 \tau - k \zeta)}, \quad m \in \mathbb{N}, n \in \mathbb{Z} \bmod 2m\mathbb{Z}. \quad (4.130)$$

Taking into account the transformations of η and θ under the generators S and T of the modular group (cf. [354] (13.6.8))

$$S(\zeta, \tau) = (\zeta/\tau, -1/\tau), \quad T(\zeta, \tau) = (\zeta, \tau + 1), \quad (4.131)$$

$$\eta(\tau) = e^{\pi i \tau / 12} \prod_{n \in \mathbb{N}} (1 - q^n), \quad \eta(-1/\tau) = \sqrt{-i\tau} \eta(\tau), \quad \eta(\tau + 1) = e^{\pi i / 12} \eta(\tau), \quad (4.132)$$

$$\begin{aligned} (\theta_{1,2} - \theta_{-1,2})(\zeta/\tau, -1/\tau) &= -i\sqrt{-i\tau}(\theta_{1,2} - \theta_{-1,2})(\zeta, \tau), \\ (\theta_{1,2} - \theta_{-1,2})(\zeta, \tau + 1) &= e^{\pi i / 4}(\theta_{1,2} - \theta_{-1,2})(\zeta, \tau), \end{aligned} \quad (4.133)$$

we can easily see that

$$\mathcal{M}(\zeta, \tau) \equiv \left| \frac{\eta(\tau)}{\theta_{1,2}(\zeta, \tau) - \theta_{-1,2}(\zeta, \tau)} \right|^2 \quad (4.134)$$

is a modular invariant for the case $m_0 = 0$, $\Lambda_0 = -\rho$ (cf. (4.88), (4.120)).

Next one should try to generalize the Weyl group W_a (and then $W_a^\pm$, $W_e^\pm$) for an arbitrary affine Kac–Moody algebra $\mathcal{G}$. From the paper of Malikov, Feigin, and Fuchs [436] we know that the singular vectors of Theorem 5 are related to the second order Casimir operator Ω of $sl(2, \mathbb{C})$ by renormalizing the Virasoro algebra Sugawara-type construction of [357, 354]. The operator Ω for the finite-dimensional algebra $\mathcal{G}_0$ underlying $\mathcal{G}$ will give rise to similar singular vectors and we should be able to define the group W_a in an analogous way. However, there should be similar singular vectors [436] related to the higher order Casimir operator Ω^i (whose number together with Ω is equal to $\ell \equiv \text{rank } \mathcal{G}_0$). Thus we should end up with Weyl groups W_a^i , $i = 1, \dots, \ell$, each of which is isomorphic to W_a . Moreover they should be mutually commuting since the Casimir operators commute.

Another path is pursued in [164] and Section 3.5. There we show that the group W_a is also a Weyl group for the Virasoro and $N = 1$ super-Virasoro algebras. This is not

trivial, since these algebras do not possess a (generalized) Cartan matrix. The notion of V -active elements of W_a plays a crucial role in these considerations.

Finally, one should try to apply these ideas to superalgebras with associated (generalized) Cartan matrices. We have already noted in [188], [159] (see also Volume 3 and here Section 4.4), that if a Verma module over such a superalgebra is reducible w. r. t. some odd root α (such that $(\alpha, \alpha) = 0$) then the corresponding odd embeddings between Verma modules cannot be described by the Weyl–Kac group for the same reasons as in the case of singular Verma modules over affine Lie algebras. Consequently there we have introduced the so-called *odd reflections* (or translations) by a formula looking exactly as (4.19).

4.3 Special representation of the $SO(3, 2)$ Kac–Moody algebra

4.3.1 Introduction

Representations of the compact Kac–Moody Lie algebras play an important role in building conformal field theories in two dimensions which in turn can be used as building blocks for constructing string theories, cf. e. g. [268]. They provide the exact solution to the Wess–Zumino–Witten model describing string moving in a group manifold [270]. The coset construction of Goddard–Kent–Olive [276] for any coset G/H , where G is a compact group and H is a subgroup, provides a wider class of conformal field theories. Given a representation of an affine G , one can impose H invariance to project down to a smaller representation which can be used to describe a string moving in spaces that are not necessarily group manifolds. The affine algebras used in these constructions are compact because they have been used to describe the internal sector of the string theory.

A natural question to ask is whether the representations of noncompact affine Lie algebras may also have a role to play in constructing conformal field theories. Later, Dixon–Peskin–Lykken [151] examined this issue for affine $SO(2, 1)$. They found the interesting result that all unitary irreps of the $N = 2$ superconformal algebra for $c > 3$ may be obtained from irreps of affine $SO(2, 1)$. The unitarity of the irrep is achieved by a $U(1)$ projection. (Another projection was discussed in [38].) It is clearly of interest to see what we can learn from the representations of more general noncompact groups. Bars [42] has discussed some aspects of the irreps based on G/H for a general noncompact group G , and examined in more detail the cosets $SU(1, 1)/U(1)$, $SU(N, M)/SU(N) \times SU(M) \times U(1)$ and $SL(2, C)/SU(2)$. Here we focus our attention on the coset $\mathcal{M} \equiv SO(3, 2)/SO(3) \times SO(2)$ with the following motivations in mind.

In the case of $SO(2, 1)/U(1)$ construction, the $U(1)$ is connected in a nontrivial way with the automorphism group of the $N = 2$ superconformal algebra in two dimensions.

Taking this as a hint, we expect that in a coset construction based on $\mathcal{M}$, the subalgebra $\mathrm{SO}(3) \oplus \mathrm{SO}(2) \cong \mathrm{SU}(2) \oplus U(1)$ may be related to the automorphism group of a particular class of $N = 4$ superconformal algebras in two dimensions.

Our second motivation is that the irreps based on the coset $\mathcal{M}$ may provide the ingredients that may be needed in the construction of a string theory where the symmetry of spacetime is the anti de Sitter group $\mathrm{SO}(3, 2)$ rather than the Poincaré group.

In constructing the irreps corresponding to a coset G/H one has to start from an irrep of the finite-dimensional group G . Our third motivation for considering the coset $\mathcal{M}$ is that $\mathrm{SO}(3, 2)$ has truly remarkable representations known as singletons, which were first discovered by Dirac in 1963 [148]. These representations have been extensively studied by Flato–Fronsdal [246, 230, 231, 232], and Evans [211]. There are two singleton representations, called Di and Rac. In terms of the lowest eigenvalue E_0 of the $\mathrm{SO}(2)$ which has the interpretation of energy, and the spin s_0 of the state with this energy, the Di has $(E_0 = 1, s_0 = 1/2)$, while the Rac has $(E_0 = 1/2, s_0 = 0)$. These irreps have remarkably reduced weight spaces. Consequently, the singleton field theory has a very large gauge symmetry which enables one to gauge away the singleton fields everywhere except on the boundary of the anti de Sitter space [231]. Moreover, the direct product of two singletons decomposes into infinitely many massless states of the anti de Sitter group [230]. We want to construct the irreps of the Kac–Moody $\mathrm{SO}(3, 2)$ based on singletons to see if and how these remarkable properties carry over to the affine case.

Here we focus on the Rac. The fact that the Rac is spinless makes the analysis somewhat easier. The unitarity of the Rac representation ensures the unitarity of our representation at level 0. We ensure the unitarity of our representation at level 1 by explicitly computing the norms of the states at this level. This imposes nontrivial constraints on the structure of the representation and puts the following restrictions on the central charge, namely: $k > E_0$, $k > 2$. We make the conjecture that there are no further restrictions on the lowest weight vector $|E_0 = 1/2, s_0 = 0, k >$ coming from unitarity at higher levels. Unitarity in the region $k > 3$ has been conjectured in general [42]. However, we shall consider the more intriguing cases $2 < k < 3$. (The case $k = 3$ is excluded for reasons explained in the text.) The most interesting case turns out to be the representation with $k = 5/2$ built on the Rac. In this case the representation space (the lowest weight module, or Verma module) is highly reducible because of the presence of an infinite number of null vectors. Factoring out the subspaces (submodules) built on these null vectors reduces very much the resulting module. One of the main points of our approach is to relate these submodules by subgroups of the Weyl group, called reduced Weyl groups, and thus to diminish the number of the independent submodules to a finite number. This can be viewed as the affine analogue of the “singleton phenomenon”, i. e. the remarkably reduced nature of the weight space. We also rederive this well-known result by our methods.

4.3.2 Preliminaries

4.3.2.1 $SO(3, 2)$

The algebra $g = SO(3, 2)$ is 10-dimensional and its generators $M_{AB} = -M_{BA}$, ($A, B = 0, 1, 2, 3, 4$), $\eta_{AB} = \text{diag}(+ - - - +)$, obey

$$[M_{AB}, M_{CD}] = i(\eta_{BC}M_{AD} - \eta_{AC}M_{BD} - \eta_{BD}M_{AC} + \eta_{AD}M_{BC}). \quad (4.135)$$

Their hermiticity properties are (we follow the conventions of [472])

$$M_{AB}^\dagger = M_{AB} \quad \text{for } (A, B) = (0, 4), (a, b), \quad (4.136a)$$

$$M_{AB}^\dagger = -M_{AB} \quad \text{for } (A, B) = (0, a), (a, 4), \quad (4.136b)$$

where $a, b = 1, 2, 3$. Following [472] we shall use the following basis for g :

$$M^{\varepsilon\varepsilon'} \equiv \frac{1}{2}(M_1^\varepsilon + i\varepsilon' M_2^\varepsilon), \quad \varepsilon, \varepsilon' = \pm, \quad (4.137a)$$

$$K^\varepsilon \equiv M_3^\varepsilon, \quad \varepsilon = \pm, \quad K^3 \equiv M_{04}, \quad (4.137b)$$

$$J^{\varepsilon'} \equiv -\varepsilon' M_{23} - iM_{31}, \quad \varepsilon' = \pm, \quad J^3 \equiv M_{12}. \quad (4.137c)$$

The generators $J^\pm, J^3$ generate an $SO(3)$ subalgebra of g and together with K^3 they generate the maximal compact subalgebra $SO(3) \oplus SO(2)$ of g , while J^3, K^3 generate the compact Cartan subalgebra h of g .

Using a finite-dimensional representation of g (see e. g. [472]) and normalizing the Cartan–Killing form $(,)$ on g as $(X, Y) \equiv \frac{1}{2} \text{tr}(XY)$, we have (for the non-zero cases):

$$\begin{aligned} (M^{++}, M^{--}) &= (M^{+-}, M^{-+}) = (K^3, K^3) = (J^3, J^3) = 1 \\ (K^+, K^-) &= -(J^+, J^-) = 2. \end{aligned} \quad (4.138)$$

We note for further reference that the Cartan involution θ is defined by

$$\begin{aligned} \theta : X &\mapsto X, \quad \text{if } X \text{ is compact,} \quad (X = K^3, J^3, J^\pm), \\ \theta : X &\mapsto -X, \quad \text{if } X \text{ is noncompact.} \end{aligned} \quad (4.139)$$

We also note the hermiticity properties of the new basis,

$$\begin{aligned} (M^{\varepsilon\varepsilon'})^\dagger &= M^{-\varepsilon, -\varepsilon'}, \\ (K^\varepsilon)^\dagger &= K^{-\varepsilon}, \quad (K^3)^\dagger = K^3, \\ (J^{\varepsilon'})^\dagger &= -J^{-\varepsilon'}, \quad (J^3)^\dagger = J^3. \end{aligned} \quad (4.140)$$

4.3.2.2 The affine $\mathrm{SO}(3, 2)$ algebra

The affinization of $g = \mathrm{SO}(3, 2)$ is done in the standard manner. The affine Lie algebra $\tilde{g}$ is given by

$$\tilde{g} = g \otimes \mathbb{R}[t, t^{-1}] \oplus \mathbb{R}\hat{c} \oplus \mathbb{R}\hat{d} \quad (4.141)$$

where $\mathbb{R}[t, t^{-1}]$ is the algebra of real Laurent polynomials in t , $\hat{c}$ is in the centre of $\tilde{g}$, and $\hat{d} = t \frac{d}{dt}$ has the properties of $-L_0$ (L_0 is the zero mode generator in the Virasoro algebra).

The commutation relations are

$$[X \otimes t^m, Y \otimes t^n] = [X, Y] \otimes t^{m+n} + m\hat{c}\delta_{m,-n}(X, Y), \quad (4.142a)$$

$$[\hat{d}, X \otimes t^m] = mX \otimes t^m, \quad (4.142b)$$

where, $X, Y \in g$, $m, n \in \mathbb{Z}$. For convenience we set the central element $\hat{c} = k$, $k \in \mathbb{R}$, and denote $X_n \equiv X \otimes t^n$. The Cartan subalgebra of $\tilde{g}$ is $(h \equiv h \otimes 1)$

$$\tilde{h} = h \oplus \mathbb{R}\hat{c} \oplus \mathbb{R}\hat{d}. \quad (4.143)$$

The Cartan involution θ and the hermitian conjugation are extended to $\tilde{g}$ as follows:

$$\theta X_n = (\theta X)_{-n}, \quad \theta \hat{c} = \hat{c}, \quad \theta \hat{d} = \hat{d}, \quad (4.144)$$

$$(X_n)^\dagger = (X^\dagger)_n, \quad \hat{c}^\dagger = \hat{c}, \quad \hat{d}^\dagger = \hat{d}. \quad (4.145)$$

4.3.2.3 Root systems

By the definition of a Cartan subalgebra h the elements of the adjoint action $ad H$ can be diagonalized simultaneously in g . For $\alpha \in h^*$, where h^* is the dual space of linear functionals over h , let us define

$$g_\alpha \equiv \{X \in g : [H, X] = \alpha(H)X \text{ for all } H \in h\}. \quad (4.146)$$

The set $\Delta \equiv \{\alpha \in h^* : \alpha \neq 0, g_\alpha \neq 0\}$ is called the *root system* of g relative to h . Every root system has a special basis, the so-called system Δ_S of *simple* roots, so that every root of Δ can be expressed as a linear combination of the elements of Δ_S with integer coefficients of the same sign. Then the roots for which the coefficients are positive (respectively, negative) are called positive (respectively, negative) roots; then $\Delta^+(\Delta^-)$ denotes the set of positive (negative) roots and if $\alpha \in \Delta^+$, then $-\alpha \in \Delta^-$.

Further we shall use the decomposition

$$g = g_+ \oplus h \oplus g_-, \quad g_\pm \equiv \bigoplus_{\alpha \in \Delta^\pm} g_\alpha, \quad (4.147)$$

where $\dim g_\alpha = 1$. Explicitly for $g = SO(3, 2)$ we have

$$\begin{aligned} \Delta^\pm = \{ \pm\alpha_1 \pm \alpha_2 \pm \alpha_3, \pm\alpha_4 : \alpha_1(K^3, J^3) = (0, 1), \alpha_2(K^3, J^3) = (1, 1), \\ \alpha_3(K^3, J^3) = (1, 0), \alpha_4(K^3, J^3) = (1, 1) \}. \end{aligned} \quad (4.148)$$

The simple roots are $\Delta_S = \{\alpha_1, \alpha_2\}$, while

$$\alpha_3 = \alpha_1 + \alpha_2, \quad \alpha_4 = 2\alpha_1 + \alpha_2. \quad (4.149)$$

Let us denote the root space vector of g_α by X_α . Then, up to a constant, we have

$$X_{\pm\alpha_1} = J^\pm, \quad X_{\pm\alpha_2} = M^{\pm\mp}, \quad X_{\pm\alpha_3} = K^\pm, \quad X_{\pm\alpha_4} = M^{\pm\pm}. \quad (4.150)$$

The simple root system Δ_S is also a basis for h^* . To every root α there corresponds an element $H_\alpha \in h$ which we choose as follows [150]:

$$H_\alpha \equiv [X_\alpha, X_{-\alpha}] / (X_\alpha, X_{-\alpha}). \quad (4.151)$$

Denoting $H_i \equiv H_{\alpha_i}$, we have

$$H_1 = J^3, \quad H_2 = K^3 - J^3, \quad H_3 = K^3 = H_1 + H_2, \quad H_4 = K^3 + J^3 = 2H_1 + H_2. \quad (4.152)$$

Finally, we can use the duality to define a scalar product in h^* by setting

$$(\alpha, \beta) \equiv (H_\alpha, H_\beta). \quad (4.153)$$

Explicitly for $g = SO(3, 2)$ we have for the non-zero products

$$(\alpha_1, \alpha_1) = 1 = (\alpha_3, \alpha_3), \quad (\alpha_2, \alpha_2) = (\alpha_4, \alpha_4) = 2, \quad (4.154a)$$

$$(\alpha_1, \alpha_2) = -1, \quad (\alpha_1, \alpha_4) = (\alpha_2, \alpha_3) = (\alpha_3, \alpha_4) = 1. \quad (4.154b)$$

We shall also need the root system $\Delta^{\mathbb{C}}$ of the complexification $g^{\mathbb{C}}$ of g . However, Δ and $\Delta^{\mathbb{C}}$ may be identified since g is maximally split [305].

Next the root system $\tilde{\Delta}$ of the affine $SO(3, 2)$ algebra is given by [354]

$$\tilde{\Delta} = \{n\bar{d} + \alpha : n \in \mathbb{Z}, \alpha \in \Delta\} \cup \{n\bar{d} : n \in \mathbb{Z} \setminus \{0\}\} \quad (4.155)$$

where $\bar{d} \in \tilde{h}$ is the dual of $\hat{d}$, $\bar{d}(\hat{d}) = 1$, $\bar{d}(H) = 0$, $H \in h$, $\bar{d}(\hat{c}) = 0$. The system of simple roots is $\tilde{\Delta}_S = \{\alpha_0\} \cup \Delta_S$, $\alpha_0 = \bar{d} - \tilde{\alpha}$, where $\tilde{\alpha}$ is the highest root of Δ ($\tilde{\alpha}$ is characterized by the fact that $\tilde{\alpha} + \alpha \notin \Delta$ for $\forall \alpha \in \Delta^+$). In our case $\tilde{\alpha} = \alpha_4$, $\alpha_0 = \bar{d} - \alpha_4$. The scalar product (4.153) is extended to $\tilde{d}$ by setting $(\tilde{d}, \alpha) = 0$, $\forall \alpha \in \Delta$. Thus $(\alpha_0, \alpha) = -(\alpha_4, \alpha)$, $\forall \alpha \in \Delta$.

4.3.3 Lowest weight representations of g and $\tilde{g}$

4.3.3.1 Defining relations

Lowest weight representations of a simple Lie algebra or a Kac–Moody algebra g' are characterized by their lowest weight $\Lambda \in h'^*$, and a lowest weight vector $|\Lambda\rangle$ so that

$$H|\Lambda\rangle = \Lambda(H)|\Lambda\rangle, \quad H \in h', \quad (4.156a)$$

$$X|\Lambda\rangle = 0, \quad X \in g'_-. \quad (4.156b)$$

For $g = \text{SO}(3, 2)$ we denote $E_0 = \Lambda(K^3)$, $m_0 = \Lambda(J^3)$, $|\Lambda\rangle = |E_0, m_0\rangle$ and we have

$$K^3|E_0, m_0\rangle = E_0|E_0, m_0\rangle, \quad J^3|E_0, m_0\rangle = m_0|E_0, m_0\rangle, \quad (4.157a)$$

$$X|E_0, m_0\rangle = 0, \quad X = M^{\pm}, K^-, J^-. \quad (4.157b)$$

The eigenvalue of K^3 is called the energy, since upon contraction of $\text{SO}(3, 2)$ to the Poincaré algebra, K^3 goes over to the translation operator P_0 . Analogously, J^3 is the third component of the angular momentum. Thus E_0 and m_0 are the lowest values of the energy and third component of angular momentum j , respectively; note that $m_0 \leq 0$. This is why we deal with lowest weight representations, since we would like the energy to be bounded from below. These representations of $\text{SO}(3, 2)$ will be denoted by $D(E_0, s_0)$, $s_0 = -m_0 \geq 0$.

In the affine case $|\Lambda\rangle = |N, E_0, s_0, K\rangle$, where N is the level, and k is the central charge. Thus,

$$\begin{aligned} K_0^3|\Lambda\rangle &= E_0|\Lambda\rangle, \quad J_0^3|\Lambda\rangle = m_0|\Lambda\rangle, \\ \hat{d}|\Lambda\rangle &= -N|\Lambda\rangle, \quad \hat{c}|\Lambda\rangle = k|\Lambda\rangle. \end{aligned} \quad (4.158)$$

It is natural to choose $N = 0$ for the lowest weight. The conditions (4.157) are now enlarged to be

$$X_n|E_0, s_0, k\rangle = 0, \quad n > 0, \quad \forall X \in g, \quad (4.159a)$$

$$X_0|E_0, s_0, k\rangle = 0, \quad X = M^{\pm}, K^-, J^-. \quad (4.159b)$$

4.3.3.2 Unitarity

The representation space $\mathbf{H}$ of the lowest weight representations has as its basis the monomials

$$(M^{++})^{k_1}(M^{+-})^{k_2}(K^+)^{k_3}(J^+)^{k_4}|E_0, s_0\rangle, \quad k_1, k_2, k_3, k_4 = 0, 1, 2, \dots, \quad (4.160)$$

for g , while in the $\tilde{g}$ case the monomials are

$$\begin{aligned} &(M^{++})^{k_1}(M_{-n'_1}^{--})^{k'_1}(M_{-n'_2}^{+-})^{k_2}(M_{-n'_2}^{-+})^{k'_2}(K_{-n'_3}^+)^{k_3}(K_{-n'_3}^-)^{k'_3}(J_{-n'_4}^+)^{k_4}(J_{-n'_4}^-)^{k'_4} \\ &\times (K_{-n'_5}^3)^{k'_5}(J_{-n'_6}^3)^{k'_6}|E_0, s_0, k\rangle, \quad k_i, k'_i, n_i = 0, 1, \dots, n'_i = 1, 2, \dots \end{aligned} \quad (4.161)$$

(One can choose a different order of the factors in (4.160) and (4.161), but for each representation this order should be fixed). The scalar product in $\mathbf{H}$ is given by a contravariant hermitian form $\phi(u, v)$, $u, v \in \mathbf{H}$ [354] such that

$$\phi(Xu, v) = \phi(u, (\theta X)^+ v), \quad X \in g \text{ or } \bar{g}. \quad (4.162)$$

Unitarity of the representation is equivalent to the positivity of the norm of each non-zero vector $v \in \mathbf{H}$, i. e.

$$\|v\| \equiv \phi(u, v)0, \quad v \neq 0. \quad (4.163)$$

One assumes that the lowest weight vector has norm equal to 1. The unitary lowest weight representations of g are as follows [148, 246, 211]. In terms of (E_0, s_0) , $s_0 = -m_0$ (s_0 is the spin of the representation) they are

$$\text{Rac} : (E_0, s_0) = \left(\frac{1}{2}, 0\right), \quad \text{Di} : (E_0, s_0) = \left(1, \frac{1}{2}\right), \quad (4.164)$$

$$\left(E_0 > \frac{1}{2}, s_0 = 0\right), \quad \left(E_0 > 1, s_0 = \frac{1}{2}\right), \quad (E_0 \geq s_0 + 1, s_0 \geq 1). \quad (4.165)$$

The first two are the singleton representations and the last ones for $E_0 = s_0 + 1$ correspond to the spin- s_0 massless representations.

Let us consider the representations of the SO(3, 2) Kac–Moody algebra built on unitary representations of SO(3, 2). We first compute the simplest norms:

$$\|M_{-n}^{\varepsilon\varepsilon'}|\Lambda\rangle\| = kn + \varepsilon E_0 - \varepsilon' s_0 \quad (n = 0 \Rightarrow \varepsilon = +), \quad (4.166a)$$

$$\|K_{-n}^{\varepsilon}|\Lambda\rangle\| = 2(kn + \varepsilon E_0) \quad (n = 0 \Rightarrow \varepsilon = +), \quad (4.166b)$$

$$\|J_{-n}^{\varepsilon'}|\Lambda\rangle\| = 2(\varepsilon' s_0 - kn), \quad (n = 0 \Rightarrow \varepsilon' = +), \quad (4.167a)$$

$$\|K_{-n}^3|\Lambda\rangle\| = -kn, \quad (4.167b)$$

$$\|J_{-n}^3|\Lambda\rangle\| = -kn, \quad n > 0. \quad (4.167c)$$

We consider modules built on the unitary irreps of SO(3, 2) listed in (4.164) and (4.165), i. e. $E_0 \geq 1/2 + s_0$ or $E_0 \geq s_0 + 1$. Then it is clear that the norms (4.166) and (4.167) cannot be all nonnegative: For $k \geq E_0 + s_0$ the norms (4.166) are positive but the norms (4.167) are negative, (or zero in (4.167a) for $n = 0 = s_0$), while for $k \leq -s_0$ the norms (4.167) are nonnegative but the norms (4.166) are negative (except for $n = 0$). We should get rid of the negative norm states. To this end, following [151], we choose to work with $k > 0$. With this choice, all states in (4.167) have nonpositive norms. Eliminating them is equivalent to keeping only the states $|x\rangle$ annihilated by the conjugates of the generators creating negative norm states, i. e.

$$K_n^3|x\rangle = 0, \quad n > 0, \quad J_1^\pm|x\rangle = 0, \quad J_1^3|x\rangle = 0. \quad (4.168)$$

It follows that $J_n^\pm|x\rangle = 0 = J_n^3|x\rangle$, $n > 1$.

For simplicity we shall consider the spin zero case, $s_0 = 0$. Then $\|J_0^+|\Lambda\rangle\| = 0$; moreover, as we shall see below, this is a null state and it decouples from the representation. Thus we shall impose also the condition

$$J_0^+|\Lambda\rangle = 0. \quad (4.169)$$

Following [151] we consider the states at level 1. It is not difficult to see that (when (4.169) holds) the level 1 states are given by

$$S_1 = M_{-1}^-|x\rangle, \quad (4.170)$$

$$S_2 = K_{-1}^-|x\rangle,$$

$$S_3 = M_{-1}^+|x\rangle,$$

$$S_4 = (\alpha J_{-1}^- + \beta M_{-1}^- K_0^+ + \gamma K_{-1}^- M_0^{+-})|x\rangle,$$

$$S_5 = (\alpha' J_{-1}^+ + \beta' M_{-1}^+ K_0^+ + \gamma' K_{-1}^- M_0^{++})|x\rangle,$$

$$S_6 = (\alpha_1 K_{-1}^3 + \beta_1 J_{-1}^3 + \gamma_1 M_0^{--} + \delta_1 M_{-1}^- M_0^{+-} + \varepsilon_1 K_{-1}^- K_0^+)|x\rangle,$$

$$S_7 = [\alpha_2 K_{-1}^+ + \beta_2 M_{-1}^- K_0^+ M_0^{++} + \gamma_2 M_{-1}^- K_0^+ M_0^{+-} + \delta_2 K_{-1}^- (K_0^+)^2 \\ + \varepsilon_2 J_{-1}^+ M_0^{+-} + \eta_2 J_{-1}^- M_0^{++}]|x\rangle,$$

$$S_8 = [\alpha_3 M_{-1}^{+-} + \beta_3 M_{-1}^- (K_0^+)^2 + \gamma_3 M_{-1}^- M_0^{++} M_0^{+-} + \delta_3 M_{-1}^- (M_0^{+-})^2 \\ + \varepsilon_3 J_{-1}^- K_0^+ + \eta_3 K_{-1}^- K_0^+ M_0^{+-} + \lambda_3 K_{-1}^3 M_0^{+-} + \kappa_3 J_{-1}^3 M_0^{+-}]|x\rangle,$$

$$S_9 = [\alpha_4 M_{-1}^{++} + \beta_4 M_{-1}^+ (K_0^+)^2 + \gamma_4 M_{-1}^+ M_0^{++} M_0^{+-} + \delta_4 M_{-1}^+ (M_0^{++})^2 \\ + \varepsilon_4 J_{-1}^+ K_0^+ + \eta_4 K_{-1}^- K_0^+ M_0^{++} + \lambda_4 K_{-1}^3 M_0^{++} + \kappa_4 J_{-1}^3 M_0^{++}]|x\rangle,$$

where $\alpha, \beta, \gamma, \dots$ are constants, and $|x\rangle$ is any state involving only the generators $M_0^{\pm\pm}$, K_0^+ , i. e. $|x\rangle$ would be a state of the unitary irrep $D(E_0, 0)$ if we replace $|E_0, 0, k\rangle$ by $|E_0, 0\rangle$.

Furthermore, it is enough to consider $|x\rangle = |\Lambda\rangle$. The first three states fulfil (4.168). Since these are to be the admissible states, their norm should be positive or zero; in the latter case they should decouple from the system. Their norms (as seen from (4.166)) are all proportional to $k - E_0$ with positive coefficients. However, $k - E_0 = 0$ is not possible because, as we shall see below, S_2 is not a null vector for this choice. Thus the positivity of the norms of S_1, S_2, S_3 gives

$$k > E_0. \quad (4.171)$$

From S_4 there are two independent states obeying (4.168):

$$S_{4a} = (E_0 J_{-1}^- + K_{-1}^- M_0^{+-})|\Lambda\rangle, \quad (4.172a)$$

$$S_{4b} = (M_{-1}^- K_0^+ + K_{-1}^- M_0^{+-})|\Lambda\rangle, \quad (4.172b)$$

with norms

$$\|S_{4a}\| = 2E_0 k(k-1)(k-E_0), \quad (4.173a)$$

$$\|S_{4b}\| = 4E_0(k-E_0). \quad (4.173b)$$

Thus we obtain $k \geq 1$. However, we shall obtain stronger conditions below. The sector S_5 is obtained from S_4 by changing $J^- \rightarrow J^+$, $M^{\pm-} \rightarrow M^{\pm+}$. There are two states obeying (4.168) analogous to those given in (4.172) and their norms are given by the right hand side of (4.173). Therefore no new restrictions arise in this sector.

In the sector S_6 there are three independent states obeying (4.168):

$$S_{6a} = (2E_0 K_{-1}^3 + k K_{-1}^- K_0^+) |\Lambda\rangle, \quad (4.174a)$$

$$S_{6b} = (E_0 (K_{-1}^3 - J_{-1}^3) + k M_{-1}^{-1} M_0^{+-}) |\Lambda\rangle, \quad (4.174b)$$

$$S_{6c} = (E_0 (K_{-1}^3 + I_{-1}^3) + k M_{-1}^{--} M_0^{++}) |\Lambda\rangle, \quad (4.174c)$$

with norms

$$\|S_{6a}\| = 2kE_0(k-1)(k-E_0), \quad (4.175)$$

$$\|S_{6b}\| = \|S_{6c}\| = E_0 k(k-2)(k-E_0). \quad (4.176)$$

Thus we obtain

$$k > 2. \quad (4.177)$$

($k = 2$ is not possible because it gives zero norm states which do not decouple.) The sectors S_7 , S_8 and S_9 have five independent states each. For lack of space we do not give them here; in any case their norms are positive if (4.171) and (4.177) hold.

To summarize, the unitarity conditions are (4.171) and (4.177). We shall make the plausible conjecture that there are no further restrictions on $|\Lambda\rangle$ coming from unitarity at higher levels. There is an exceptional value of k , namely $k = 3$, or for arbitrary non-compact algebras g , $k = h^\vee$, where $h^\vee$ is the dual Coxeter number of $g^\mathbb{C}$, the complexification of g . Unitarity in the region $k > h^\vee$ is conjectured in general [42]. However, the region $k < h^\vee$ is more interesting, since, as we shall see, there occur many null states. Their elimination reduces the representation space and is similar to the reduction of the singleton representation space.

4.3.4 Null vectors and reduced Weyl groups

Verma modules and their reducibility

While (4.171) and (4.177) guarantee nonnegativity of the scalar products at level 1, there are many null states, i. e. nonzero states with zero norms, which must be eliminated. A systematic way to find all null states is to find the null vectors (singular vectors) of the Verma modules V^Λ [150] with the same lowest weight as $\mathbf{H}$. As is well known, a Verma module V^Λ is the lowest weight module with lowest weight vector v_0 induced from the one-dimensional representation of a Borel subalgebra $b = h^\mathbb{C} \oplus g_-^\mathbb{C}$, of $g^\mathbb{C}$, where $g_-^\mathbb{C}$, $h^\mathbb{C}$, $g^\mathbb{C}$ are the complexifications of g_- , h , g . For our purposes we can identify V^Λ with

the representation space $\mathbf{H}$ which has the basis (4.160) or (4.161). (This is possible, since $\mathfrak{g}$ is maximally split.)

A singular vector v_s of a Verma module V^Λ is defined as follows:

$$Xv_s = 0, \quad X \in \mathfrak{g}_-, \quad Hv_s = \Lambda'(H)v_s, \quad \forall H \in \mathfrak{h}, \quad (4.178)$$

$v_s \neq v_0$, $\Lambda' \neq \Lambda$. Thus v_s can be represented as (see [150] and [172]):

$$v_s = \mathcal{P}(\mathfrak{g}_+)|\Lambda\rangle, \quad (4.179a)$$

$$[H, \mathcal{P}(\mathfrak{g}_+)] = (\Lambda' - \Lambda)(H)\mathcal{P}(\mathfrak{g}_+), \quad \forall H \in \mathfrak{h}, \quad (4.179b)$$

where $\mathcal{P}(\mathfrak{g}_+)$ is a polynomial.

Furthermore a singular vector of a Verma module exists iff the Bernstein–Gel’fand–Gel’fand (BGG) [70] criterion for finite-dimensional $\mathfrak{g}$, or the Kac–Kazhdan [355] criterion for affine $\tilde{\mathfrak{g}}$, is fulfilled. It is the condition

$$2(\Lambda - \rho, \alpha) = -m(\alpha, \alpha), \quad m = 1, 2, \dots, \quad (4.180)$$

where $\rho \in \mathfrak{h}^*$ is defined by the condition that

$$\rho(a_i^\vee) = 1, \quad a_i^\vee \equiv \frac{2\alpha_i}{(\alpha_i, \alpha_i)}, \quad \forall \alpha_i \in \Delta_S. \quad (4.181)$$

For finite-dimensional $\mathfrak{g}$, ρ_0 can be defined as the half-sum of all positive roots; for $\text{SO}(3, 2)$ we have

$$\rho_0 = \frac{1}{2}(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) = 2\alpha_1 + \frac{3}{2}\alpha_2. \quad (4.182)$$

If $\tilde{\mathfrak{g}}$ is the affinization of $\mathfrak{g}$ then we have (see [354] and [172])

$$\tilde{\rho} = \rho_0 + \hbar^\vee \bar{c}, \quad (4.183)$$

where $\bar{c} \in \tilde{\mathfrak{h}}$ is the dual of $\hat{c}$, $\bar{c}(\hat{c}) = 1$, $\bar{c}(H) = 0$, $H \in \mathfrak{h}$, and $\bar{c}(\vec{d}) = 0$. The scalar product (4.153) is extended to $\bar{c}$ by setting $(\bar{c}, \alpha) = 0$, $\forall \alpha \in \Delta$, $(\bar{c}, \vec{d}) = 1$. (In general, $\tilde{\rho}$ may contain a term proportional to $\vec{d}$, since the basis of $\tilde{\mathfrak{h}}^*$ consists of the basis of $\mathfrak{h}$, $\bar{c}$, and $\vec{d}$. However, because of the degeneracy of the Cartan matrices of affine Kac–Moody algebras, the coefficient of $\vec{d}$ cannot be determined from (4.181) and being arbitrary may be set to zero.)

Before applying this to our situation we need an explicit expression for Λ . This expression is

$$\Lambda = (E_0 - s_0)\alpha_1 + E_0\alpha_2 + k\bar{c}. \quad (4.184)$$

This expression follows from (4.156a), (4.158), (4.148) and the above definition of $\bar{c}$. The term $-N\vec{d}$ in the expression for Λ is missing, since we have taken the natural value for the lowest weight, namely $N = 0$.

Next we substitute in (4.180) the expressions for Λ and $\bar{\rho}$; we also substitute α with the real positive roots from (4.155), i. e. $\alpha = n\bar{d} \pm \alpha_i$, $n = 0, 1, \dots$, taking for $n = 0$ only the plus sign is possible. Let us denote by $m_i^\pm$ the resulting m from (4.180). Thus we see that V^Λ is reducible in the affine case, if at least one of the following numbers is a positive integer:

$$m_1^\pm = 2n(3 - k) \pm (1 + 2s_0), \quad (4.185a)$$

$$m_2^\pm = n(3 - k) \pm (1 - E_0 - s_0), \quad (4.185b)$$

$$m_3^\pm = 2n(3 - k) \pm (3 - 2E_0), \quad (4.185c)$$

$$m_4^\pm = n(3 - k) \pm (2 - E_0 + s_0). \quad (4.185d)$$

We do not consider reducibility with respect to imaginary roots $\alpha = n\bar{d}$ (with $(\alpha, \alpha) = 0$), which is possible only for $k = 3$. One reason is that the Sugawara construction of L_0 in terms of the generators of $\tilde{g}$ involves $k - 3$ in the denominator. Thus we exclude $k = 3$.

Note that the BGG conditions for $g = SO(3, 2)$ are obtained by setting $n = 0$ and taking the only possible sign, plus, in (4.185).

In the analysis of the previous section we used the claim that S_1 given in (4.170,) with $|x\rangle = |\Lambda\rangle$, is a null (singular) vector for $s_0 = 0$. This means that $m_1^+ = 1$ should hold for $n = 0$ and $s_0 = 0$, and it indeed does. We further claimed that there can be no singular vectors in the sectors S_2, S_6 , with $k = E_0$, $k = 2$, respectively. Indeed, to have singular vectors in the sector S_2 , we should have $m_3^- = 1$, for $n = 1$, which is not the case ($m_3^- = 3$). A singular vector in the sector S_6 is possible only for the imaginary roots $\alpha = n\bar{d}$ and $k = 3$, and is not possible for $k = 2$.

Going back to the analysis of (4.185) we see that for central charge $k > 3$ there would be very few reducibility conditions fulfilled. Thus we choose to analyze the interval $2 < k < 3$.

Further we observe that we get most reducibility conditions for the representations of $\tilde{g}$ for the special value of $k = 5/2$, built on the Rac representation of g , i. e. for

$$E_0 = \frac{1}{2}, \quad s_0 = 0, \quad k = \frac{5}{2}. \quad (4.186)$$

Indeed, substituting (4.186) in (4.185) we have

$$m_1^\pm = n \pm 1, \quad m_2^\pm = \frac{n \pm 1}{2}, \quad m_3^\pm = n \pm 2, \quad m_4^\pm = \frac{n \pm 3}{2}. \quad (4.187)$$

Thus the Verma module V^Λ with $|\Lambda\rangle = |\frac{1}{2}, 0, \frac{5}{2}\rangle$ is reducible with respect to the following roots:

$$n\bar{d} + \alpha_1, \quad n \geq 0, \quad m_1^+ = n + 1, \quad (4.188a)$$

$$n\bar{d} - \alpha_1, \quad n \geq 2, \quad m_1^- = n - 1, \quad (4.188b)$$

$$n\bar{d} + \alpha_2, \quad n = 2t + 1, \quad t \geq 0, \quad m_2^+ = t + 1, \quad (4.188c)$$

$$n\bar{d} - \alpha_2, \quad n = 2t + 1, \quad t \geq 1, \quad m_2^- = t, \quad (4.188d)$$

$$n\bar{d} + \alpha_3, \quad n \geq 0, \quad m_3^+ = n + 2, \quad (4.188e)$$

$$n\bar{d} - \alpha_3, \quad n \geq 3, \quad m_3^- = n - 2, \quad (4.188f)$$

$$n\bar{d} + \alpha_4, \quad n = 2t + 1, \quad t \geq 0, \quad m_4^+ = t + 2, \quad (4.188g)$$

$$n\bar{d} - \alpha_4, \quad n = 2t + 1, \quad t \geq 2, \quad m_4^- = t - 1, \quad (4.188h)$$

i. e. only the roots $\bar{d} - \alpha_i$, $i = 1, 2, 3, 4$, $2\bar{d} - \alpha_3$, $3\bar{d} - \alpha_4$, $2t\bar{d} \pm \alpha_2$, $2t\bar{d} \pm \alpha_4$ are excluded.

Furthermore, we shall need the explicit expression for the weight Λ' in (4.178) and (4.179) when (4.180) is fulfilled. This expression is given by [150, 354, 172]

$$\Lambda' = \Lambda + m\alpha. \quad (4.189)$$

The reducibility of the Verma module V^Λ means that it has submodules isomorphic to other Verma modules $V^{\Lambda'}$, where Λ' is given in (4.189) with $\alpha = n\bar{d} \pm \alpha_i$, $m = m_i^\pm$ from (4.188). Fortunately only a finite number of these submodules are independent. All others are themselves submodules of this finite set.¹ To find the independent submodules we need the technique of the reduced Weyl groups [153, 154]. The idea is that the redundant submodules are related by elements of the Weyl group to the independent submodules. This is done in the next subsection.

Reduced Weyl groups

Let us recall that each root $\alpha \in \Delta$, defines a reflection s_α in h^* by the formula

$$s_\alpha(\lambda) = \lambda - (\lambda, \alpha^\vee)\alpha, \quad \alpha^\vee = 2\alpha/(\alpha, \alpha), \quad \lambda \in h^*, \quad s_\alpha^2 = 1. \quad (4.190)$$

As is well known, these reflections generate a group, called the Weyl group W of g (or $\tilde{g}$) [150, 354]. [The Weyl groups of a real Lie algebra g and its complexification differ, unless g is maximally split as $SO(3, 2)$]. W has a finite (infinite) number of elements if g is finite- (infinite-) dimensional. Actually W is generated by the reflections by simple roots $w_i = s_{\alpha_i}$, which for the affine $SO(3, 2)$ obey (besides $\omega_i^2 = 1$)

$$(w_0 w_1)^4 = (w_0 w_2)^2 = (w_1 w_2)^4 = 1. \quad (4.191)$$

For $g = SO(3, 2)$, $|W| = 8$ and all elements are

$$W = \{1, w_1, w_2, w_1 w_2, w_2 w_1, w_1 w_2 w_1, w_2 w_1 w_2, w_1 w_2 w_1 w_2\}. \quad (4.192)$$

¹ If V^Λ were reducible with respect to all simple roots (here $\alpha_1, \alpha_2, \alpha_0 = \bar{d} - \alpha_4$) then only the (three) submodules corresponding to these reducibilities would be relevant.

We shall not write down the explicit parametrization of the full Weyl group $\tilde{W}$ for $\tilde{g}$, but only the elements which correspond to the roots in (4.188):

$$s_{n\bar{d}+\alpha_1} = (w_1 w_2 w_0)^{2n} w_1, \quad (4.193a)$$

$$s_{n\bar{d}-\alpha_1} = (w_0 w_2 w_1)^{2n-1} w_0 w_2, \quad (4.193b)$$

$$s_{(2t+1)\bar{d}+\alpha_2} = (w_2 w_1 w_0 w_1)^{2t+1} w_2, \quad (4.193c)$$

$$s_{(2t+1)\bar{d}-\alpha_2} = (w_1 w_0 w_1 w_2)^{2t} w_1 w_0 w_1, \quad (4.193d)$$

$$s_{n\bar{d}+\alpha_3} = (w_2 w_1 w_0)^{2n} w_2 w_1 w_2, \quad (4.193e)$$

$$s_{n\bar{d}-\alpha_3} = (w_0 w_1 w_2)^{2(n-1)} w_0 w_1 w_0, \quad (4.193f)$$

$$s_{(2t+1)\bar{d}+\alpha_4} = (w_1 w_2 w_1 w_0)^{2t+1} w_1 w_2 w_1, \quad (4.193g)$$

$$s_{(2t+1)\bar{d}-\alpha_4} = (w_0 w_1 w_2 w_1)^{2t} w_0. \quad (4.193h)$$

Actually, the above expressions are valid also for some of the roots not listed in (4.188), namely for $\bar{d} - \alpha_i$ ($i = 1, 2, 3, 4$), $2\bar{d} - \alpha_3$ and $3\bar{d} - \alpha_4$. For the other roots we have

$$s_{2t\bar{d}+\alpha_2} = (w_2 w_1 w_0 w_1)^{2t} w_2, \quad (4.194a)$$

$$s_{(2t+2)\bar{d}-\alpha_2} = (w_1 w_0 w_1 w_2)^{2t+1} w_1 w_0 w_1, \quad (4.194b)$$

$$s_{2t\bar{d}+\alpha_4} = (w_1 w_2 w_1 w_0)^{2t} w_1 w_2 w_1, \quad (4.194c)$$

$$s_{(2t+2)\bar{d}-\alpha_4} = (w_0, w_1, w_2 w_1)^{2t+1} w_0. \quad (4.194d)$$

Our problem is that we do not have the whole affine Weyl group at our disposal since α_2 and $\alpha_0 = \bar{d} - \alpha_4$ are not in the list (4.188). Thus we look at subgroups of the Weyl group, which we call reduced Weyl groups [153, 154]. They should be generated by elements s_α with α in the list (4.188). The natural thing to try is to take Weyl groups of affine Lie subalgebras of $\tilde{g}$. We have found it most convenient to work with *four* subalgebras of $\tilde{g}$, each isomorphic to the affine $SL(2)$, i. e. $A_1^{(1)}$. Their systems of simple roots are

$$\{\alpha_1, 2\bar{d} - \alpha_1\}, \quad \{\bar{d} + \alpha_2, 3\bar{d} - \alpha_2\}, \quad \{\alpha_3, 3\bar{d} - \alpha_3\}, \quad \{\bar{d} + \alpha_4, 5\bar{d} - \alpha_4\}. \quad (4.195)$$

These roots are the lowest level roots of each type in (4.188). The corresponding reduced Weyl groups W_R^i , $i = 1, 2, 3, 4$, are isomorphic to the well-known Weyl group of $A_1^{(1)}$, discussed in detail in Section 4.1.4. In terms of the generating elements they are explicitly given as follows (for fixed i):

$$u_{0s} \equiv (\tilde{w}'_i \tilde{w}_i)^s, \quad u_{1s} \equiv (\tilde{w}_i \tilde{w}'_i)^s, \quad u'_{0s} \equiv u_{0s} \tilde{w}'_i, \quad u'_{1s} \equiv u_{1s} \tilde{w}_i, \quad s \geq 0, \quad (4.196)$$

with

$$\tilde{w}_1 = s_{\alpha_1} = w_1, \quad \tilde{w}'_1 = s_{2\bar{d}-\alpha_1} = (w_0 w_2 w_1)^3 w_0 w_2,$$

$$\begin{aligned}
\tilde{w}_2 &= s_{\bar{d}+\alpha_2} = w_2 w_1 w_0 w_1 w_2, & \tilde{w}'_2 &= s_{3\bar{d}-\alpha_2} = (w_1 w_0 w_1 w_2)^2 w_1 w_0 w_1, \\
\tilde{w}_3 &= s_{\alpha_3} = w_2 w_1 w_2, & \tilde{w}'_3 &= s_{3\bar{d}-\alpha_3} = (w_0 w_1 w_2)^4 w_1 w_0 w_1, \\
\tilde{w}_4 &= s_{\bar{d}+\alpha_4} = w_1 w_2 w_1 w_0 w_1 w_2 w_1, & \tilde{w}'_4 &= s_{5\bar{d}-\alpha_4} = (w_0 w_1 w_2 w_1)^4 w_0.
\end{aligned} \quad (4.197)$$

(For fixed i the element $\tilde{w}_i$ corresponds to w_0 from Section 4.1.4; cf. e. g. formula (4.31), while the element $\tilde{w}'_i$ corresponds to w_1 there.)

Furthermore, it will be convenient to introduce a shifted action of $w \in W$ by the formula

$$w * \Lambda \equiv w(\Lambda - \rho) + \rho. \quad (4.198)$$

Using this, (4.190) and (4.180) we can rewrite (4.189) as follows:

$$\Lambda' = \Lambda + m\alpha = \Lambda - (\Lambda - \rho, \alpha^\vee)\alpha = s_\alpha * \Lambda. \quad (4.199)$$

The reduced Weyl group W_R^i guarantees that all submodules which are Verma modules with weights $\Lambda' = w * \Lambda$, with $w \in W_R^i$, are submodules of the two Verma modules $V^{\Lambda'}$ with $w = \tilde{w}_i$ and $w = \tilde{w}'_i$. For those elements of (4.193) which are not in (4.195) this check should be done independently.²

Furthermore, we shall use (analogously to (4.31)) also the notation $\Lambda_{0s} \equiv u_{0s} * \Lambda$, $\Lambda_{1s} \equiv u_{1s} * \Lambda$, $\Lambda'_{0s} \equiv u'_{0s} * \Lambda$, $\Lambda'_{1s} \equiv u'_{1s} * \Lambda$, and the notation $V_{0s} \equiv V^{\Lambda'}$, with $\Lambda' = \Lambda_{0s}$, and similarly the notation V_{1s} , V'_{0s} and V'_{1s} , (cf. the figure in (4.29)).

Now all elements in (4.193a) with $n = 2\ell$, $\ell \geq 0$, are of the form u'_{1s} , $s = \ell$, i. e.

$$s_{2\ell\bar{d}+\alpha_1} = (w_1 w_0 w_2)^{4\ell} w_1 = (w_1 \tilde{w}'_1)^\ell w_1 = u'_{1\ell}, \quad \ell \geq 0. \quad (4.200)$$

This means, as we said above in general, that $V'_{1\ell}$, $\ell \geq 1$, are submodules of V'_{10} and of V'_{00} . For the elements in (4.193a) with $n = 2\ell + 1$, $\ell \geq 0$, we have

$$\begin{aligned}
s_{(2\ell+1)\bar{d}+\alpha_1} &= (w_1 w_0 w_2)^{4\ell+2} w_1 = w_1 \cdot w_0 w_2 w_1 w_0 w_2 \cdot (w_1 w_0 w_2)^{4\ell} w_1 \\
&= w_1 \cdot w_0 w_2 w_1 w_0 w_2 \cdot u'_{1\ell}.
\end{aligned} \quad (4.201)$$

The question is whether $\tilde{V}'_{1\ell} \equiv V^{\Lambda'}$, with $\Lambda' = s_{(2\ell+1)\bar{d}+\alpha_1}$, is a submodule of $V'_{1\ell}$, and hence of V'_{10} and of V'_{00} . For this we check that $V'_{1\ell}$ is reducible with respect to $\bar{d} - \alpha_1$, and that there is a submodule of $V'_{1\ell}$, namely, $\tilde{V}''_{1\ell} \equiv V^{\Lambda''}$, with $\Lambda'' = \tilde{\Lambda}''_{1\ell} \equiv s_{\bar{d}-\alpha_1} * \Lambda'_{1\ell} = w_0 w_2 w_1 w_0 w_2 * \Lambda'_{1\ell} = (w_1 w_0 w_2)^{4\ell+2} * \Lambda$. Next we check that $\tilde{V}'_{1\ell}$ is reducible with respect to α_1 , so that $\tilde{V}'_{1\ell}$ is a submodule of $\tilde{V}''_{1\ell}$ since $\tilde{\Lambda}'_{1\ell} = w_1 * \tilde{\Lambda}''_{1\ell}$. Thus indeed $\tilde{V}'_{1\ell}$ is submodule of V'_{10} and V'_{00} .

² The careful reader will notice that in (4.196) there are also elements which are not in (4.193) and (4.194), namely, u_{0s} and u_{1s} . This is a partial case of the general fact that a Weyl group contains many more elements besides those of the form s_α , $\alpha \in \Delta$.

Next we consider the elements in (4.193b). Those with $n = 2\ell + 2, \geq 0$, are of the form $u'_{0s}, s \geq \ell$, i. e.

$$s_{(2\ell+2)\bar{d}-\alpha_1} = (w_0 w_2 w_1)^{4\ell+3} w_0 w_2 = (\tilde{w}'_i \tilde{w}_i)^\ell \tilde{w}'_i = u'_{0\ell}, \quad \ell \geq 0. \quad (4.202)$$

Thus $V'_{0\ell}, \ell \geq 1$, are submodules of V'_{10} and V'_{00} . For the elements in (4.193b) with $n = 2\ell + 3, \ell \geq 0$, we have

$$\begin{aligned} s_{(2\ell+3)\bar{d}-\alpha_1} &= (w_0 w_2 w_1)^{4\ell+5} w_0 w_2 = w_0 w_2 w_1 w_0 w_2 \cdot w_1 \cdot (w_0 w_2 w_1)^{4\ell+3} w_0 w_2 \\ &= w_0 w_2 w_1 w_0 w_2 \cdot w_1 \cdot u'_{0\ell} = w_0 w_2 w_1 w_0 w_2 \cdot u_{0\ell}. \end{aligned} \quad (4.203)$$

Thus we have only to check that $V_{0\ell}$ is reducible with respect to $\bar{d} - \alpha_1$, so that $\tilde{V}_{0\ell} \equiv V^{\Lambda'}$ with $\Lambda' = \tilde{\Lambda}_{0\ell} \equiv s_{(2\ell+3)\bar{d}-\alpha_1} * \Lambda$, is a submodule of V'_{10} and V'_{00} . [Note that $\tilde{V}'_{0\ell} \equiv V^{\Lambda'}$, with $\Lambda' = \tilde{\Lambda}'_{0\ell} \equiv w_1 * \tilde{\Lambda}_{0\ell}$ is a submodule of $\tilde{V}_{0\ell}$, but this is not important for our purposes as $\tilde{\Lambda}'_{0\ell} = w * \Lambda$, for $w \neq s_\alpha, \forall \alpha \in \Delta$. Also $\tilde{V}'_{0\ell}$ is a submodules of $\tilde{V}'_{0\ell}$ and $\tilde{V}''_{0\ell}$ is a submodule of $\tilde{V}_{0\ell}$, which is also not important for our purposes.]

Thus all submodules corresponding to (4.188a,b) and (4.193a,b) will be taken care of if we factor the submodules generated by $s_{\alpha_1} = w_1, s_{2\bar{d}-\alpha_1} = \tilde{w}'_1 = (w_0 w_2 w_1)^3 w_0 w_2$.

We have treated the cases (4.188c,d), (4.188e,f), and (4.188g,h) in a similar fashion to the case described above.

Thus we are left with eight embeddings. Next, we also notice that $\tilde{w}_4 = w_1 \tilde{w}_2 w_1$ and that $V^{\Lambda'}$, with $\Lambda' = w'_1 * \Lambda$, $V^{\Lambda''}$ with $\Lambda'' = \tilde{w}_2 * \Lambda'$, are reducible with respect to $\bar{d} + \alpha_2, \alpha_1$, respectively. Thus $V^{\Lambda'''}$ with $\Lambda''' = \tilde{w}_4 * \Lambda$ is a submodule of $V^{\Lambda'}$. One should also consider the decomposition $\tilde{w}'_4 = w_1 s_{5\bar{d}-\alpha_2} w_1$. Indeed, $V^{\Lambda'}$, with $\Lambda' = w'_1 * \Lambda$ is reducible with respect to $s_{5\bar{d}-\alpha_2}$. However, $V^{\Lambda''}$ with $\Lambda'' = \tilde{s}_{5\bar{d}-\alpha_2} * \Lambda'$, is reducible with respect to $4\bar{d} + \alpha_1$, and not with respect to α_1 .

Thus there are seven relevant submodules $V^{\Lambda'}$, with weights $\Lambda' = w * \Lambda$, with $w = w_1, \tilde{w}'_1, \tilde{w}_2, \tilde{w}'_2, \tilde{w}_3, \tilde{w}'_3, \tilde{w}_4$. The cases $w = w_1$ and $w = \tilde{w}_2 = w_2 w_1 w_2$ are inherited from the singleton. The corresponding singular vectors are $J_0^+ |\Lambda\rangle$ and $(K_0^+)^2 + 4M_0^{+-} M_0^{++} |\Lambda\rangle$. To factor them away we have to impose

$$\begin{aligned} v_1^+ &= J_0^+ |\Lambda\rangle = 0, \quad |\Lambda\rangle = |1/2, 0, 5/2\rangle, \\ v_3^+ &= [(K_0^+)^2 + 4M_0^{+-} M_0^{++}] |\Lambda\rangle = 0, \end{aligned} \quad (4.204)$$

subsuming (4.169). The other singular vectors are at levels 1, 2, 3 and 5. We give only the leading terms:

$$v_1^- = (J_-^- + \cdots) |\Lambda\rangle, \quad \alpha = 2\bar{d} - \alpha_1, \text{ level 2}, \quad (4.205a)$$

$$v_2^+ = (M_{-1}^{+-} + \cdots) |\Lambda\rangle, \quad \alpha = \bar{d} + \alpha_2, \text{ level 1}, \quad (4.205b)$$

$$v_2^- = (M_{-3}^{+-} + \cdots) |\Lambda\rangle, \quad \alpha = 3\bar{d} - \alpha_2, \text{ level 3}, \quad (4.205c)$$

$$v_3^- = (K_{-3}^- + \cdots) |\Lambda\rangle, \quad \alpha = 3\bar{d} - \alpha_3, \text{ level 3}, \quad (4.205d)$$

$$v_4^- = (M_{-5}^{+-} + \cdots) |\Lambda\rangle, \quad \alpha = 5\bar{d} - \alpha_4, \text{ level 5}. \quad (4.205e)$$

4.3.5 Summary

Our representation space now can be described as follows. It has the basis (4.161) with lowest weight:

$$E_0 = 1/2, \quad s_0 = 0, \quad k = 5/2. \quad (4.206)$$

The states which survive are those which obey the conditions (4.168):

$$K_n^3|x\rangle = 0, \quad n > 0, \quad J_1^\pm|x\rangle = 0, \quad J_1^3|x\rangle = 0, \quad (4.207)$$

and those which are not annihilated when we impose conditions (4.204) and (4.205):

$$v_1^+ = v_3^+ = v_1^- = v_2^+ = v_2^- = v_3^- = v_4^- = 0, \quad (4.208)$$

on the vacuum.

4.4 Multiplets of Verma modules over the $\mathfrak{osp}(2, 2)^{(1)}$ super Kac–Moody algebra

The infinite-dimensional superalgebra $\mathfrak{osp}(2, 2)^{(1)}$ is interesting for several reasons. The underlying finite-dimensional superalgebra $\mathfrak{osp}(2, 2)$ is used in the building of two-dimensional $N = 2, 4$ conformal supergravities coupled to nonlinear σ -models [63, 475]. The superalgebra $\mathfrak{osp}(2, 2)$ is also a subalgebra of the infinite-dimensional $N = 2$ Neveu–Schwarz-type superalgebra [1] (considered in Section 3.7). It is expected that $\mathfrak{osp}(2, 2)$ and $N = 2$ Neveu–Schwarz superalgebras will have similarities in their representation theories, especially in the structure of their Verma modules. This expectation is based on other examples of pairs of infinite-dimensional (super-) Lie algebras, namely, Virasoro– $A^{(1)}$, $N = 1$ Neveu–Schwarz– $\mathfrak{osp}(1, 2)^{(1)}$, which share $\mathfrak{sl}(2, \mathbb{C})$, $\mathfrak{osp}(1, 2)$, respectively, as subalgebras; cf. Section 4.1, [155, 156]. Finally, this paper is a natural extension to the super Kac–Moody case of our programme of multiplet classification of Verma modules over finite-dimensional (super-) Lie algebras, affine Lie algebras, (super-) Virasoro algebras, quantum groups, considered in earlier sections of this volume and in the first three volumes of our tetralogy. Because of the lack of space we consider only the two most interesting examples: Verma modules with non-singular integral highest weights and the singular Verma modules.

4.4.1 Definitions and notation

The $\mathfrak{osp}(2, 2)$ superalgebra $\mathcal{G}$ [350] is a complex Lie superalgebra with even generators $K_0, K_{\pm 1}, Y_0$, odd generators $Q_{\pm 1/2}, \bar{Q}_{\pm 1/2}$, and super-Lie brackets:

$$[K_n, K_m] = (m - n)K_{n+m}, \quad [Y_0, K_0] = 0, \quad (4.209)$$

$$\begin{aligned}
 [K_n, Q_s] &= \left(s - \frac{9}{2}\right) Q_{n+s}, & [K_n, \bar{Q}_s] &= \left(s - \frac{9}{2}\right) \bar{Q}_{n+s}, \\
 [Q_r, \bar{Q}_s]_+ &= (r-s)Y_{r+s} - 2K_{r+s}, & [Q_r, Q_s]_+ &= 0 = [\bar{Q}_r, \bar{Q}_s]_+, \\
 [Y_0, Q_r] &= Q_r, & [Y_0, \bar{Q}_r] &= -\bar{Q}_r.
 \end{aligned}$$

We have the decomposition $\mathcal{G} = \mathcal{G}^+ \oplus \mathcal{H} \oplus \mathcal{G}^-$, where $\mathcal{H}$ is the Cartan subalgebra spanned by K_0, Y_0 , while $\mathcal{G}^\pm$ are spanned by $L_{\pm 1}, Q_{\pm 1/2}, \bar{Q}_{\pm 1/2}$, respectively. The positive roots of the system $\Delta = \Delta(\mathcal{G}, \mathcal{H})$ are $\Delta^+ = \Delta_0^+ \cup \Delta_1^+$, where $\Delta_0^+ = \{\alpha_3 = \frac{1}{2}\delta\}$, $\Delta_1^+ = \{\alpha_1 = \delta + \epsilon, \alpha_2 = \delta - \epsilon\}$, are the even, respectively, odd, positive roots, so that $\delta(L_0, Y_0) = (1/2, 0)$, $\epsilon(L_0, Y_0) = (0, 1)$, and obviously $\Delta_1^+ = \Delta_S$, the system of simple roots. The bilinear form of $\mathcal{H}^*$ is fixed by $\langle \delta, \delta \rangle = 1/2 = -\langle \epsilon, \epsilon \rangle$, $\langle \delta, \epsilon \rangle = 0$. The canonical generators E_i, F_i, H_i , $i = 1, 2, 3$, are chosen as

$$\begin{aligned}
 E_1 &= Q_{1/2}, & F_1 &= \bar{Q}_{-1/2}, \\
 [E_1, F_1]_+ &= \langle E_1, F_1 \rangle H_1 = -2H_1 = Y_0 - 2K_0,
 \end{aligned} \tag{4.210a}$$

$$\begin{aligned}
 E_2 &= \bar{Q}_{1/2}, & F_2 &= Q_{-1/2}, \\
 [E_2, F_2]_+ &= \langle E_2, F_2 \rangle H_2 = -2H_2 = -Y_0 - 2K_0,
 \end{aligned} \tag{4.210b}$$

$$\begin{aligned}
 E_3 &= K_1, & F_3 &= K_{-1}, \\
 [E_3, F_3] &= \langle E_3, F_3 \rangle H_3 = -H_3 = -2K_0 = -H_1 - H_2
 \end{aligned} \tag{4.210c}$$

$$\langle H_i, H_j \rangle = \langle \alpha_i, \alpha_j \rangle, \quad \langle H_i, X \rangle = \alpha_i(X), \quad \forall X \in \mathcal{H}. \tag{4.210d}$$

(The Cartan–Killing form $\langle X, Y \rangle$, $X, Y \in \mathcal{G}$, defined in (4.210) is consistent with the super-trace $\text{str } XY$ if we introduce a matrix realization for $\mathcal{G}$.)

Let $\tilde{\mathcal{G}} = \mathcal{G} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\hat{c} \oplus \mathbb{C}\hat{d}$ be the super Kac–Moody algebra $\text{osp}(2, 2)^{(1)}$ corresponding to $\mathcal{G}$ [348], where $\mathbb{C}[t, t^{-1}]$ is the algebra of Laurent polynomials in t , $\mathbb{C}\hat{c}$ is the centre of $\tilde{\mathcal{G}}$, $\hat{d} \equiv t \, d/dt$, and the super-Lie brackets are

$$[X \otimes t^m, Z \otimes t^n] = \{X, Z\} \otimes t^{m+n} + m\hat{c}\delta_{m,-n}\langle X, Z \rangle, \tag{4.211a}$$

$$[\hat{d}, X \otimes t^m] = mX \otimes t^m, \tag{4.211b}$$

where $\{X, Z\} = [X, Z]_+$ if both X, Z are odd, otherwise $\{X, Z\} = [X, Z]$. Then $\tilde{\mathcal{H}} = \mathcal{H} \oplus \mathbb{C}\hat{c} \oplus \mathbb{C}\hat{d}$ is the Cartan subalgebra of $\tilde{\mathcal{G}}$. Let $\tilde{\Delta} = \tilde{\Delta}^+ \cup \tilde{\Delta}^-$ be the root system $(\tilde{\mathcal{G}}, \tilde{\mathcal{H}})$, then

$$\tilde{\Delta}^+ = \{\beta_{n\kappa\alpha} = n\bar{d} + \kappa\alpha, \mid n \in \mathbb{N}, \kappa = 0, \pm 1; n = 0, \kappa = \pm 1; \alpha \in \Delta^+\} \tag{4.212}$$

where $\bar{d} \in \mathcal{H}^*$ is defined by $\bar{d}(\hat{d}) = 1$, $\bar{d}(\hat{c}) = 0 = \bar{d}(X)$, $X \in \mathcal{H}$. Then $\tilde{\Delta}_S = \{\alpha_0 = \bar{d} - \tilde{\alpha}, \alpha_1, \alpha_2\}$ is the system of simple roots, where $\tilde{\alpha} = \alpha_3$ is the highest root of Δ^+ . The weights will be denoted by Λ :

$$\Lambda = c\bar{c} + d\bar{d} + \Lambda_0 \in \tilde{\mathcal{H}}^*, \quad c, d \in \mathbb{C}, \quad \Lambda_0 = \lambda_1\alpha_1 + \lambda_2\alpha_2 \in \mathcal{H}^* \tag{4.213}$$

where $\hat{c}$ is defined by $\hat{c}(c) = 1$, $\hat{c}(d) = 0 = \hat{c}(X)$, $X \in \mathcal{H}$. Then we extend $\langle, \rangle$ to $\tilde{\mathcal{H}}^*$ which is clear for $\mathcal{H}^*$ and for $\bar{c}$, $\bar{d}$ is given by $\langle \bar{c}, \bar{d} \rangle = 1$, $\langle \bar{c}, \bar{c} \rangle = \langle \bar{d}, \bar{d} \rangle = \langle \bar{c}, \mu \rangle = \langle \bar{d}, \mu \rangle = 0$, $\mu \in \mathcal{H}^*$. Thus the product of two roots in our notation is

$$\langle \beta_{n\kappa\alpha}, \beta_{n'\kappa'\alpha'} \rangle = \kappa\kappa' \langle \alpha, \alpha' \rangle. \quad (4.214)$$

The roots $\tilde{\Delta}^+$ are divided into *imaginary roots*, $\kappa = 0$, $n > 0$; *real even roots*, $\kappa \neq 0$, $\alpha = \alpha_3$, and *real odd roots*, $\kappa \neq 0$, $\alpha = \alpha_1, \alpha_2$. The canonical generators of $\tilde{\mathcal{G}}$ are chosen as $e_0 = F_3 \otimes t$, $f_0 = E_3 \otimes t^{-1}$, $h_0 = \hat{c} - H_3 = \hat{c} - 2K_0 \otimes 1 = [e_0, f_0]$, $(e_i, f_i, h_i) \equiv (E_i, F_i, H_i) \otimes 1$, $i = 1, 2$.

The element important for the representation theory, $\tilde{\rho} = \rho_c \bar{c} + \rho_d \bar{d} + \rho \in \tilde{\mathcal{H}}^*$, $\rho \in \mathcal{H}^*$, is defined by $(\tilde{\rho}, \alpha_i) = \frac{1}{2}(\alpha_i, \alpha_i)$, $i = 0, 1, 2$, to give $\tilde{\rho} = 2\bar{c} + \rho' \bar{d}$, where ρ' is arbitrary and we set it equal to zero. (The finite-dimensional part is defined as $\rho = \rho_0 - \rho_1$, where $2\rho_0 = \sum_{\alpha \in \Delta_0^+} 2\delta$, $2\rho_1 = \sum_{\alpha \in \Delta_1^+} 2\delta$, to give $\rho = 0$.) Note that the coefficient of $\bar{c}$, here 2, is called the dual Coxeter number; see, e. g. [354].

4.4.2 Verma modules, their reducibility and multiplets

We record the standard decomposition $\tilde{\mathcal{G}} = \tilde{\mathcal{G}}^+ \oplus \tilde{\mathcal{H}} \oplus \tilde{\mathcal{G}}^-$, $\tilde{\mathcal{G}}^\pm = \sum_{\beta \in \tilde{\Delta}^\pm} \tilde{\mathcal{G}}_\beta^\pm$, where $\tilde{\mathcal{G}}_\beta^\pm$ is the root space corresponding to the root $\pm\beta$. A highest weight module (HWM) over $\tilde{\mathcal{G}}$ is characterized by its weight, $\Lambda \in \tilde{\mathcal{H}}^*$, and a highest weight vector v such that $Xv = 0$, $X \in \tilde{\mathcal{G}}^+$, $Xv = \Lambda(X)v$, $X \in \tilde{\mathcal{H}}$. The Verma module V^Λ is the largest HWM with highest weight Λ . We have $V^\Lambda \cong U(\tilde{\mathcal{G}}^-)v$, where $U(\tilde{\mathcal{G}}^-)$ is the universal enveloping algebra of $\tilde{\mathcal{G}}^-$. We combine the results of Kac [351] for superalgebras and of Kac–Kazhdan [355] for affine Lie algebras to see that the Verma module V^Λ is reducible if and only if one of the following relations is satisfied:

$$2\langle \Lambda + r, \beta \rangle = m\langle \beta, \beta \rangle, \quad \beta \in \tilde{\Delta}^+, \quad m \in \mathbb{N}. \quad (4.215)$$

Using $\beta = \beta_{n\kappa\alpha}$, $\Lambda = c\bar{c} + d\bar{d} + \lambda_1\alpha_1 + \lambda_2\alpha_2$, $\rho = 2\bar{c}$, and (4.214) we obtain from (4.215)

$$n(2 + c) + \kappa(\lambda_1 + \lambda_2) = m \in \mathbb{N}, \quad \text{for real even roots:}$$

$$\beta_{n\kappa\alpha_3} : n > 0, \kappa = \pm 1 \quad \text{or} \quad n = 0, \kappa = 1; \quad (4.216a)$$

$$n(2 + c) + \kappa\lambda_{2,1} = 0, \quad \text{for real odd roots}$$

$$\beta_{n\kappa\alpha_{1,2}} : n > 0, \kappa = \pm 1 \quad \text{or} \quad n = 0, \kappa = 1; \quad (4.216b)$$

$$2 + c = 0, \quad \text{for imaginary roots: } \beta_{n0} = n\bar{d}. \quad (4.216c)$$

As in [154] and earlier sections here we divide the reducible Verma modules into nonsingular Verma modules, for which $c + 2 \neq 0$, and singular Verma module, for which $c + 2 = 0$. For a singular Verma module (4.216a,b) reduce to the three conditions

$$\kappa(\lambda_1 + \lambda_2) = m, \quad \lambda_2 = 0, \quad \lambda_1 = 0, \quad (4.217)$$

only two of which may hold for a fixed Λ . Note that (4.217) with $\kappa = 1$ are the reducibility conditions for the $\text{osp}(2, 2)$ superalgebra which follow from (4.216a,b) for $n = 0$.

From [352, 355] it also follows that whenever (4.215) is fulfilled then the invariant submodule of V^Λ (realizing reducibility) is isomorphic to the Verma module $V^{\Lambda-m\beta}$, where $m = 1$ for imaginary and real odd roots. We shall depict these situations by the following diagrams:

$$V^\Lambda \xrightarrow{n, \kappa} V^{\Lambda-\beta_{n\kappa\alpha_1}}, \quad V^\Lambda \xrightarrow{n, \kappa} V^{\Lambda-\beta_{n\kappa\alpha_2}}, \quad (4.218a)$$

$$V^\Lambda \xrightarrow{m, n, \kappa} V^{\Lambda-m\beta_{n\kappa\alpha_3}}, \quad V^\Lambda \xrightarrow{n} V^{\Lambda-\beta_{n0}}. \quad (4.218b)$$

The maps in (4.218) will be called even, odd, imaginary, embedding maps, respectively, if β is a real even, real odd, imaginary, root.

It is obvious that $V^{\Lambda-\beta_{n\kappa\alpha_i}}$ in (4.218a) is also reducible and contains an invariant submodule isomorphic to $V^{\Lambda-2\beta_{n\kappa\alpha_i}}$; moreover, $V_k \equiv V^{\Lambda-k\beta_{n\kappa\alpha_i}}$, $k \in \mathbb{Z}$, is reducible and contains V_{k+1} . Thus we have an infinite chain of embedding maps. Each of these maps is realized by a singular vector v_s in the embedding module. A singular vector has the characteristics of the HWV of the embedded module: $Xv_s = 0$, $X \in \tilde{\mathcal{G}}^+$, $Xv_s = (\Lambda - m\beta)(X)$, $X \in \tilde{\mathcal{H}}$. Using the Grassmannian properties of the odd generators it can be shown that this infinite chain is exact. For example if $\beta_{n\kappa\alpha} = \alpha_1$, then $v_s = \bar{Q}_{-1/2}v$ for all V_k . Then the composition of two odd embeddings is given by $v'_s = \bar{Q}_{-1/2}v_s = \bar{Q}_{-1/2}^2v = 0$. (Thus, strictly speaking odd embeddings are not embeddings in the usual sense, since they have kernels.) The same considerations apply for $\beta_{n\kappa\alpha} = \alpha_2$, then $v_s = Q_{-1/2}v$ for all V_k . This also explains why we must set $m = 1$ for real odd roots. Finally, we should mention that the singular vectors corresponding to α_1, α_2 are the only odd singular vectors encountered in the Verma modules over $\text{osp}(2, 2)$.

It is clear that $V^{\Lambda-\beta_{n0}}$ in (4.218b) is also reducible for any imaginary root. Thus, for the purposes of the multiplet classification it will be enough to use only the elementary embedding for $\beta = \beta_{10} = \bar{d}$.

For further use we introduce the signature of $\Lambda = c\bar{c} + d\bar{d} + \lambda_1\alpha_1 + \lambda_2\alpha_2$,

$$\chi(\Lambda) \equiv [d, m + 0, \lambda_1, \lambda_2] = \{d, m_0, m_3, y\}, \quad (4.219)$$

$$m_0 \equiv \langle \Lambda + \rho, \alpha_0 \rangle = \langle \Lambda + \rho \rangle(h_0) = c + 2 - \lambda_1 - \lambda_2,$$

$$m_3 \equiv \Lambda(H_3) = \lambda_1 + \lambda_2, \quad y \equiv \Lambda(Y_0) = \lambda_1 - \lambda_2.$$

4.4.3 Verma modules with nonsingular integral highest weights

We further divide the Verma modules with nonsingular integral highest weights according to whether they are reducible with respect to some real odd root or not. We first obtain the following.

Proposition 13. *The Verma modules with nonsingular integral highest weight which are reducible only with respect to some real even roots have the signatures $\chi = \{d, m_0, m_3, y\}$, with $m_0, m_3 \in \mathbb{N}$. Each such module is grouped in a multiplet with reducible Verma modules with nonsingular integral highest weights in the following commutative diagram:*

$$(4.220)$$

where the signatures $\chi_{ak}^{(l)}$, $a = 0, 1$, $k = 0, 1, \dots$, are ($m_3 = \lambda_1 + \lambda_2 \in \mathbb{N}$, $y = \lambda_1 - \lambda_2$):

$$\chi_{0k} = \{d - k(km_0 + (k-1)m_3), m_{0k}, -m'_{0,k-1}, y\}, \quad (4.221a)$$

$$\chi'_{0k} = \{d - (k+1)((k+1)m_0 + km_3), -m_{0k}, m'_{0k}, y\}, \quad (4.221b)$$

$$\chi_{1k} = \{d - k(km_0 + (k+1)m_3), -m'_{1,k-1}, m_{1k}, y\}, \quad (4.221c)$$

$$\chi'_{1k} = \{d - k(km_0 + (k+1)m_3), m'_{1k}, -m_{1k}, y\}, \quad (4.221d)$$

$$m_{0k} \equiv (2k+1)m_0 + 2km_3, \quad m'_{0k} \equiv 2(k+1)m_0 + (2k+1)m_3$$

$$m_{1k} \equiv 2km_0 + (2k+1)m_3, \quad m'_{1k} \equiv (2k+1)m_0 + 2(k+1)m_3,$$

Proof. We consider only Verma modules reducible with respect to conditions (4.216a). Then it remains to notice that the even real roots form the root system of the affine Lie algebra $A_1^{(1)}$. Then we use the results of Section 4.1; more precisely we use (4.29), (4.30). $\square$

Proposition 14. *The Verma modules with nonsingular integral highest weight which are reducible with respect to both even and odd real roots have the signatures $\chi = [d, m_0, \lambda_1, \lambda_2]$, $d \in \mathbb{C}$, $\lambda_1, \lambda_2 \in \mathbb{Z}_+$, $\lambda_1\lambda_2 = 0$. They are grouped in multiplets of type I_1 , distinguished by a positive integer m_0 . The multiplet $I_1(m_0)$ includes all such Verma modules with signatures: ${}^1\chi^j \equiv [d, m_0 - j, j, 0]$, ${}^2\chi^j \equiv [d, m_0 - j, 0, j]$, $j = 0, 1, \dots, m_0 - 1$, ($\chi^0 = {}^1\chi^0 = {}^2\chi^0$), and also infinitely many reducible Verma modules with nonsingular integrable highest weights. Part of the diagram looks as follows:*

$$(4.222)$$

where ${}^1\chi_{10}^{j'} \equiv {}^2\chi - j$, ${}^2\chi_{10}^{j'} \equiv {}^1\chi - j$, the maps ${}^a\chi^1 \xrightarrow{1} {}^a\chi_{10}^{1'}$, $a = 1, 2$ being the compositions through χ^0 . The remaining part of the multiplet may be described as follows. First, any ${}^a\chi^j$, $j \neq 0$, module enters a submultiplet which has exactly the structure depicted in (4.220) with $m_0 \mapsto m_0 - j$, $m_3 \mapsto j$, $y = \pm j$ (for $a = 1, 2$, respectively), ${}^a\chi^j$ playing the role of $\chi = \chi_{00} = \chi_{10}$ in (4.220). The signatures of the other modules will be denoted ${}^a\chi_{0k}^j$, ${}^a\chi_{0k}^{j'}$, ${}^a\chi_{1k}^j$, ${}^a\chi_{1k}^{j'}$, and they are given by (4.221) with the above changes. Second, there

is an infinite chain starting from χ^0 (as indicated) which is obtained from (4.220) and (4.221) by setting $j = m_3 = 0$. The modules coincide two by two; the signatures of the modules in this chain are denoted and given by $\chi_{0k}^0 = {}^a\chi_{0k}^0 = {}^a\chi_{1k}^{0'}, \chi_{0k}^1 = {}^a\chi_{1k}^1 = {}^a\chi_{0,k-1}^0$. Third, the two horizontal chains in (4.222) are infinite in both directions without further intersections. Thus we consider $j \geq m_0$. For $j \neq pm_0, p \in \mathbb{N}$, there exist submultiplets with fixed $\Lambda(Y_0) = j$ as for $0 < j < m_0$; the role of χ in (4.220) is played by ${}^a\chi_{0q}^{j'}$ if $(2q - 1)m_0 < j < 2qm_0$, and by ${}^a\chi_{0q}^j$ if $2qm_0 < j < (2q + 1)m_0, q \in \mathbb{N}$. For $j = pm_0, p \in \mathbb{N}$, there exist single chains as for $j = 0$, the role of χ_0 being played by ${}^a\chi_{0q}^j = {}^a\chi_{1q}^{j'}$ for $j = (2q - 1)m_0$ and by ${}^a\chi_{1q}^j = {}^a\chi_{0,q-1}^{j'}$ for $j = 2qm_0, q \in \mathbb{N}$. This describes all members of this multiplet and all even noncomposition embedding maps. The description of all odd noncomposition embedding maps is given by the following diagrams:

$$\begin{array}{ccccccc} \dots & \nearrow & {}^1\chi_{1k}^j & \xrightarrow{k,-1} & {}^1\chi_{1k}^{j+1} & \nearrow & \dots \\ \dots & \nwarrow & {}^2\chi_{1k}^{j+1} & \xleftarrow{k,-1} & {}^2\chi_{1k}^j & \nwarrow & \dots \end{array} \quad (4.223a)$$

$$\begin{array}{ccccccc} \dots & \nwarrow & {}^1\chi_{1k}^{j'} & \xrightarrow{k,1} & {}^1\chi_{1k}^{j+1'} & \nwarrow & \dots \\ \dots & \nearrow & {}^2\chi_{1k}^{j+1'} & \xleftarrow{k,1} & {}^2\chi_{1k}^{j'} & \nearrow & \dots \end{array} \quad (4.223b)$$

$$\begin{array}{ccccccc} \dots & \nwarrow & {}^1\chi_{0k}^{j-1} & \xleftarrow{k,1} & {}^1\chi_{0k}^j & \nwarrow & \dots \\ \dots & \nearrow & {}^2\chi_{0k}^j & \xrightarrow{k,1} & {}^2\chi_{0k}^{j-1} & \nearrow & \dots \end{array} \quad (4.223c)$$

$$\begin{array}{ccccccc} \dots & \nwarrow & {}^1\chi_{0k}^{j-1'} & \xleftarrow{k,-1} & {}^1\chi_{0k}^{j'} & \nwarrow & \dots \\ \dots & \nearrow & {}^2\chi_{0k}^{j'} & \xrightarrow{k,-1} & {}^2\chi_{0k}^{j-1'} & \nearrow & \dots \end{array} \quad (4.223d)$$

The proof is easy. We only notice that the two chains of odd embedding maps may intersect only once for some $j = pm_0, p \in \mathbb{Z}_+$. $\square$

4.4.4 Verma modules with singular highest weights

Proposition 15. Any reducible singular Verma module over $\text{osp}(2, 2)^{(1)}$ belongs to a multiplet of one of the following types: S_2, S_1, S_0 . Multiplets of type S_2 are distinguished by the set $(m, \lambda), m \in \mathbb{N}, \lambda \in \mathbb{C}, \lambda^2 \neq m^2$. For a fixed such (m, λ) the singular Verma modules are arranged in the multiplet as follows:

$$(4.224)$$

where $\chi_k^s = (d - k, -m, m, \lambda)$, $\chi_k^{s'} = (d - k, m, -m, \lambda)$, $k \in \mathbb{Z}$, $d \in \mathbb{C}$. Multiplets of type S_1 are distinguished by the complex number $\lambda \neq \mathbb{Z}$. For a fixed such λ they include all singular Verma modules such that $\lambda_1 \lambda_2 = 0$, $\lambda_1 - \lambda_2 = \lambda \pmod{\mathbb{Z}}$. They are arranged in the multiplet $S_1^1(\lambda)$ as follows:

(4.225)

where ${}^1\chi_{kj}^s = [d - k, -\lambda - j, \lambda + j, 0]$, $j, k \in \mathbb{Z}$, $d \in \mathbb{C}$, and analogously for $S_1^2(\lambda)$ with ${}^1\chi_{kj}^s \mapsto {}^2\chi_{kj}^s = [d - k, -\lambda - j, 0, \lambda + j]$ and $\mapsto$ changed to $\mapsto$ in (4.225). There is only one multiplet of type S_0 . It includes all singular Verma modules such that $\lambda_1 \lambda_2 = 0$, $\lambda_1, \lambda_2 \in \mathbb{Z}$. Their signatures are given by ${}^1\chi_k^{sj} = [d - k, -j, j, 0]$, ${}^2\chi_k^{sj} = [d - k, -j, 0, j]$, ${}^1\chi_{kj}^{sj'} = {}^2\chi_k^{s, -j}$, ${}^2\chi_{kj}^{sj'} = {}^1\chi_k^{s, -j}$, $j \in \mathbb{Z}_+$, $k \in \mathbb{Z}$, $d \in \mathbb{C}$. For fixed k they are given as in (4.222) with the obvious substitutions and keeping only the fully drawn arrows. In addition we must draw the embeddings ${}^a\chi_k^{sj(l)} \circ \rightarrow {}^a\chi_{k+1}^{sj(l)}$, $k \in \mathbb{Z}$, and ${}^a\chi_k^{sj'} \xrightarrow{j, -} {}^a\chi_{k+j}^{sj}$. For all three types all even and odd noncomposition embeddings and all $\beta = \bar{d}$ imaginary embeddings are given. $\diamond$

Corollary 4. Any reducible Verma module over $\mathfrak{osp}(2, 2)$ belongs to a multiplet of type $\tilde{S}_2$, $\tilde{S}_1$, $\tilde{S}_0$. Multiplets of type $\tilde{S}_a$ are obtained from those of type S_a for $\mathfrak{osp}(2, 2)^{(1)}$ as follows. Drop the d , m_0 entries in the signatures, keep only the embeddings associated with the $\mathfrak{osp}(2, 2)$ roots $\beta = \alpha_1, \alpha_2, \alpha_3$ and keep only the modules for a fixed k . $\diamond$

Remark 6. As we have discussed in earlier chapters in the case of superalgebras we have to use odd embeddings and to extend the usual Weyl groups by odd reflections. Analogously in the case of affine algebras we use imaginary embeddings and extend the Weyl groups by imaginary reflections. Finally, in the case of affine superalgebras, as in the present case of $\mathfrak{osp}(2, 2)^{(1)}$ we use both odd and imaginary embeddings. To those correspond odd and imaginary reflections which enlarge the Weyl group in both senses. $\diamond$

Epilogue

Although our monograph spanned four volumes and more than 1300 pages there remain topics that we could not cover for various reasons. Thus in this epilogue we just mention several of those topics, giving some relevant literature.

W algebras

In a systematic study of $D = 2$ conformal quantum field theory extensions of the conformal symmetry play an important role. The algebraic structures that emerge in the study of bosonic extended symmetry are higher-spin extensions of the Virasoro algebra, which are commonly called W algebras. They were introduced by Zamolodchikov in 1985 [585]. Among the earlier papers on W algebras we mention [80, 37, 329, 497, 74, 332, 255, 81, 127, 28, 360, 141], while for the extension to supersymmetry we refer, e. g. to [87, 88, 589].

Yangians

In representation theory, a Yangian is an infinite-dimensional Hopf algebra. For any finite-dimensional semisimple Lie algebra $\mathcal{G}$ Drinfeld [197] defined an infinite-dimensional Hopf algebra $Y(\mathcal{G})$, called the Yangian of $\mathcal{G}$. This Hopf algebra is a deformation of the universal enveloping algebra of the Lie algebra of polynomial loops of $\mathcal{G}$ given by explicit generators and relations. The relations can be encoded by identities involving a rational R -matrix. Replacing it with a trigonometric R -matrix, one arrives at affine quantum groups defined in the same paper [197]. For further reading we recommend, e. g. [69, 193, 432, 454, 53, 239, 117, 198, 545, 410], in particular, for the extension to supersymmetry [87, 285, 492, 54, 258, 251, 459, 555, 304, 67].

Cluster algebras and quivers

Cluster algebras are an axiomatic class of algebras, introduced in Fomin–Zelevinsky [234]. They are constructively defined commutative rings equipped with a distinguished set of generators (cluster variables) grouped into overlapping subsets (clusters) of the same finite cardinality (the rank of the algebra in question). A cluster algebra of rank n is an integral domain A , together with some subsets of size n called clusters whose union generates the algebra A and which satisfy various conditions. We should also note that they appeared earlier in mathematical physics, though in disguise [104, 534]. In the last 15 years cluster algebras and quivers have played a

fundamental role in many different contexts in combinatorics, representation theory, geometry and mathematical physics.

A quiver is a finite oriented graph. We allow multiple edges (called arrows) but not loops (i. e. an arrow may not connect a vertex to itself) nor oriented 2-cycles (i. e. no arrows of opposite orientation may connect the same pair of vertices). A quiver does not have to be connected.

Quivers are the combinatorial data which accompany (extended) clusters and determine exchange relations between them. Mutations of quivers lie at the heart of the combinatorial framework underlying the general theory of cluster algebras.

For further reading on cluster algebras and quivers we would recommend, e. g. [235, 142, 199, 71, 312, 264, 498, 468, 438, 260, 119, 442, 279, 383, 488, 204, 368, 408, 282, 96, 118, 291, 199, 382, 511, 220].

And that's not all ...

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